Plasma Theories III

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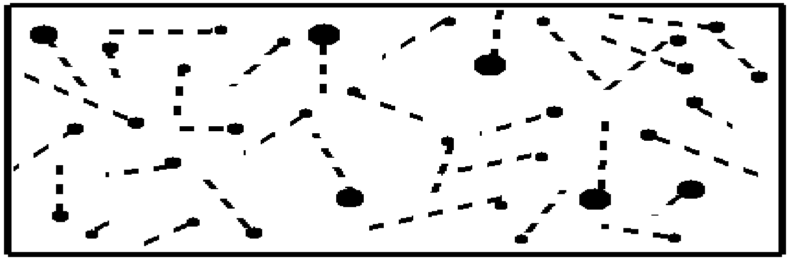
Kinetic Theory

"A theory in which the temperature of a substance increases with an increase in either the average kinetic energy of the particles or the average potential energy of separation (as in fusion) of the particles or in both when heat is added."



Within the same system all the molecules are not at the same temperature. For example, in the summer the molecules of the top layer of water are at high temperature as compared to the layer below it. Similarly, in winter the top layer of sea is ice but layers below it are at normal temperature.

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In Fluid model, we deal with all particles as a whole and also in fluid model number of particles are function of $f(r,t)$. Also, there are four variables in the fluid model

$f(x,y,z,t)$= no. of particles in the plasma system.

But in kinetic model, no. of particles are a function of $f(r,v,t)$. Also, there are seven variables in the kinetic model.

$$f(r,v,t) = \mathrm{f}(\mathrm{x},\mathrm{y},\mathrm{z},V_x,V_y,V_z,t)$$

In the fluid model we don't need the distribution function but kinetic model needs the distribution function.

So, the distribution function is defined as

$$f(r,v,t) = \frac{dN_\alpha}{d^3x d^3V} \dots\dots\dots (1)$$

$$d^3x = dx_1 dx_2 dx_3$$

$$d^3V = dV_x dV_y dV_z$$

No. density is defined as

$$dn_\alpha = \frac{dN_\alpha}{d^3x}$$

$$dN_\alpha = dn_\alpha d^3x \dots\dots\dots (2)$$

Now putting Eq.(2) in Eq.(1) we get

$$f(r,v,t) = \frac{dn_\alpha d^3x}{d^3x d^3V}$$

After cancelling d^3x we get

$$f(r,v,t) = \frac{dn_\alpha}{d^3V} \dots\dots\dots (3)$$

$$dn_\alpha = f(r,v,t)d^3V \dots\dots\dots (4)$$

Now, integrating Eq.(4)

$$n_\alpha = \int f(r,v,t)d^3V$$

$$n_\alpha = \int f(r,v,t)\, dV_x dV_y dV_z \dots\dots\dots (5)$$

In kinetic model, equation of motion is known as *Vlasov Equation*.

Vlasov Equation

Suppose $f(x, v, t)$ are the no. of particles in a system at position x , velocity u and time t. After time Δtwe have new position x^*, velocity $u^*$and time t^*.

$$x^* = x + u\Delta t$$

$$u^* = u + a\Delta t$$

$$t^* = \mathrm{t} + \Delta t$$

New positions of the particles can be defined as

$$f(x^*, u^*, t^*)$$

If we take into account the collisional term then $\frac{\partial f}{\partial t}$ exist because $\frac{\partial f}{\partial t}$ is the collisional term.

$$\left(\frac{\partial f}{\partial t}\right)_c = u\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial u} + \frac{\partial f}{\partial t}$$

Ignoring the collisional term,

$$u\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial u} + \frac{\partial f}{\partial t} = 0 \ldots\ldots\ldots (8a)$$

This equation is called the *Vlasov equation.*

The general solution is not present, only special cases.

The Properties of a Plasma

A plasma as a collection of particles

- The properties of a collection of particles can be described by specifying how many there are in a 6-dimensional volume called phase space.
  - There are 3 dimensions in “real” or configuration space and 3 dimensions in velocity space.
  - The volume in phase space is $dvdr = dv_x dv_y dv_z dxdydz$
  - The number of particles in a phase space volume is $f(\vec{r}, \vec{v}, t)dvdr$

  where f is called the distribution function.
- The density of particles of species “s” (number per unit volume)

$$n_s(\vec{r}, t) = \int f_s(\vec{r}, \vec{v}, t) dv$$

- The average velocity (bulk flow velocity)

$$\vec{u}_s(\vec{r}, t) = \int \vec{v} f_s(\vec{r}, \vec{v}, t) dv \Big/ \int f_s(\vec{r}, \vec{v}, t) dv$$

Stationary state

- Stationary state is a state independent of time.

$$\frac{\partial f}{\partial t} = 0 \qquad\qquad \frac{\partial f}{\partial t}_c = 0$$

- Then Boltzmann or Vlasov equation becomes

$$v\frac{\partial f}{\partial r} + \frac{F}{m}\frac{\partial f}{\partial v} = 0$$

- The force is related to the gradient of its potential energy

$$F = -\frac{\partial U}{\partial r}$$

$$v\frac{\partial f}{\partial r} + \frac{1}{m}\frac{-\partial U}{\partial r}\frac{\partial f}{\partial v} = 0$$

Stationary state

- The force is related to the gradient of its potential energy

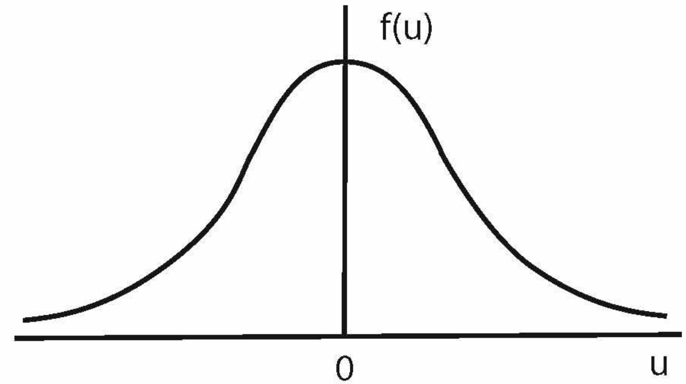
$$v\frac{\partial f}{\partial r} + \frac{1}{m}\frac{-\partial U}{\partial r}\frac{\partial f}{\partial v} = 0$$

- Integration with respect to r:

$$vf - \frac{U}{m}\frac{\partial f}{\partial v} = 0 \quad \Rightarrow \quad \frac{mvdv}{U} = \frac{df}{f} \quad \Rightarrow \quad f = f_0 \exp(\beta \frac{1}{2} m v^2)$$

- The constants are calculated from the distribution normalization.

Temperature



- The temperature is a macroscopic qua

$$< E >= \frac{\int \frac{1}{2} m v^2 f(v) dv}{\int f(v) dv}$$

- If we assume Boltzmann distribution:

$$f(v) = A \exp(-\frac{mv^2}{2k_B T})$$

- If $y^2 = \frac{mv^2}{2k_B T}$ then for one dimension
- $$< E >= \frac{k_B T \int y^2 \exp(-y^2) dy}{\int \exp(-y^2) dy} = \frac{k_B T}{2}$$
- For 3 dimensions $< E >= \frac{3k_B T}{2}$

Temperature

- The temperature is measured in Kelvin, Celsius, Fahrenheit,…
- However, in plasma community we could use the energy unit . This is not correct sometimes!!!!!!!!!!!!!

- So when we say the temperature of the plasma is 1 eV, this means that the temperature is 11605 K.

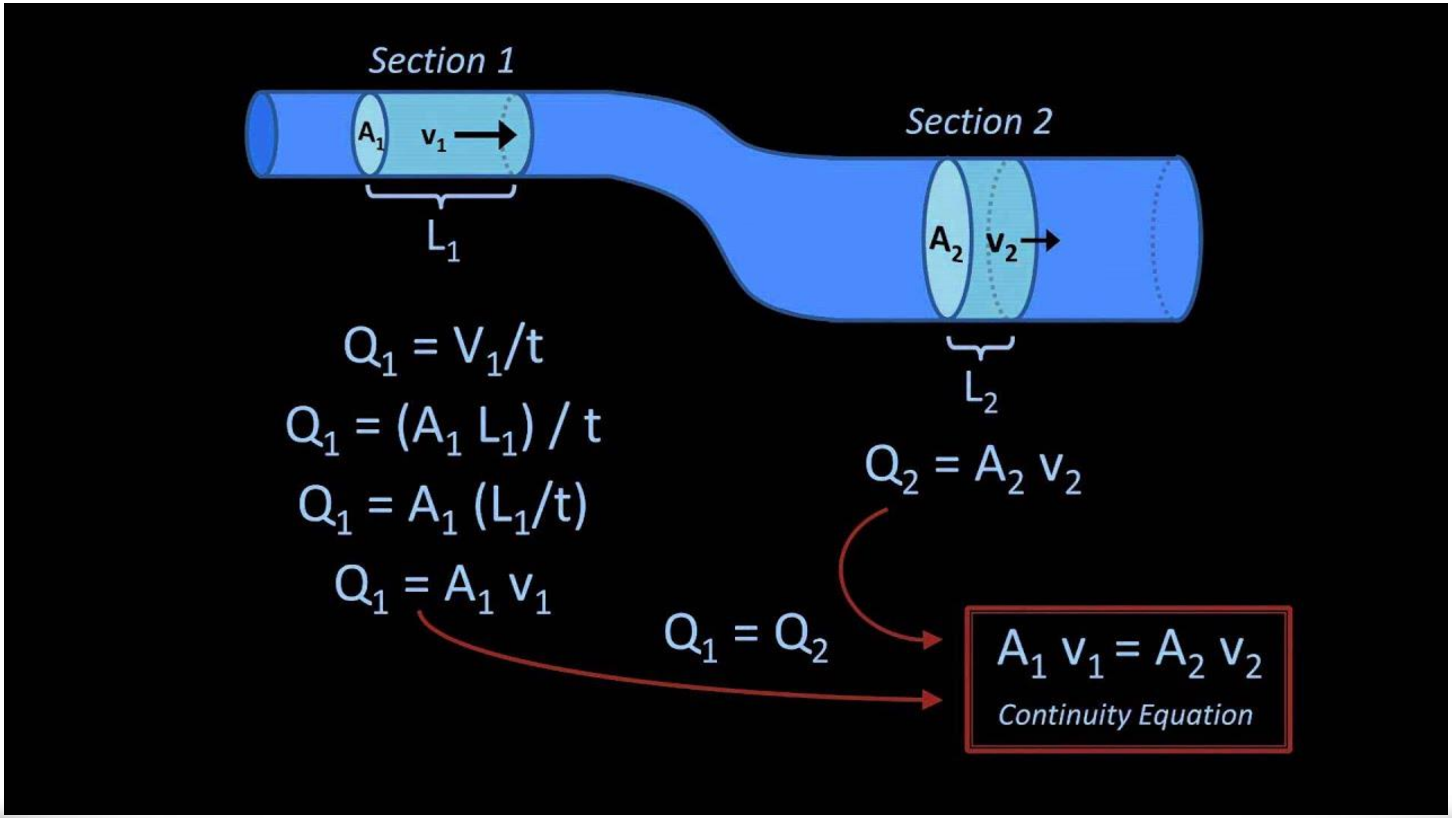
$$\mathrm{eV} = 11605K$$

- Calculate the energy of a plasma in 1D and 3 D if its temperature is 5 eV.

- In magnetized plasma, the plasma may have two temperatures, Why?

Fluid dynamics

- Continuity equations: mass, momentum, energy conservation



Moments of Boltzmann equation

- Multiply Boltzmann equation with mv^l and integrate the equation in all space
- Zero moment $l = 0$ gives contenuity equation of mass or density.
- First moment $l = 1$ gives conservation of momentum.
- Second moment $l = 2$ gives conservation of energy.

Density conservation

$$\int \frac{\partial f}{\partial t}\,d\mathbf{v} + \int (\mathbf{v}\cdot\nabla_{\mathbf{r}}) f\,d\mathbf{v}$$

$$+\frac{q}{m}\int [(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}}] f\,d\mathbf{v} = \int \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} d\mathbf{v}, \quad (4.4)$$

where we note that $d\mathbf{v} = dv_x dv_y dv_z$. We separately examine below each term of (4.4). Recalling (3.5), we can see that the first term is

$$\frac{\partial}{\partial t}\int f(\mathbf{r},\mathbf{v},t)d\mathbf{v} = \frac{\partial}{\partial t}N(\mathbf{r},t), \quad (4.5)$$

whereas the second term is

$$\begin{aligned}\int (\mathbf{v}\cdot\nabla_{\mathbf{r}}) f\,d\mathbf{v} &= \int v_x \frac{\partial}{\partial x} f\,d\mathbf{v} + \int v_y \frac{\partial}{\partial y} f\,d\mathbf{v} + \int v_z \frac{\partial}{\partial z} f\,d\mathbf{v}\\ &= \frac{\partial}{\partial x}\int v_x f\,d\mathbf{v} + \frac{\partial}{\partial y}\int v_y f\,d\mathbf{v} + \frac{\partial}{\partial z}\int v_z f\,d\mathbf{v}\\ &= \nabla_{\mathbf{r}}\cdot[N(\mathbf{r},t)\langle\mathbf{v}\rangle]\\ &= \nabla_{\mathbf{r}}\cdot[N(\mathbf{r},t)\mathbf{u}(\mathbf{r},t)], \quad (4.6)\end{aligned}$$

Continuity equation for mass or charge transport

$$\boxed{\frac{\partial}{\partial t}N(\mathbf{r},t) + \nabla_{\mathbf{r}}\cdot[N(\mathbf{r},t)\mathbf{u}(\mathbf{r},t)] = 0}, \quad (4.10)$$

Sources terms

- The average properties are governed by the basic conservation laws for mass, momentum and energy in a fluid.
- Continuity equation

$$\frac{\partial n_s}{\partial t} + \nabla \cdot n_s \vec{u}_s = S_s - L_s$$

- S_s and L_s represent sources and losses. S_s-L_s is the net rate at which particles are added or lost per unit volume.
- The number of particles changes only if there are sources and losses.

– Momentum equation

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p + \vec{J}\times\vec{B} + \rho\, F_g / m$$

where $\rho_{qs} = q_s n_s$ is the charge density, $\vec{J}_s = q_s n_s \vec{u}_s$ is the current density, and the last term is the density of non-electromagnetic forces.

– The operator $\left(\frac{\partial}{\partial t} + \vec{u}_s \cdot \nabla\right)$ is called the convective derivative and gives the total time derivative resulting from intrinsic time changes and spatial motion.

– The right side is the density of forces

- If there is a pressure gradient, then the fluid moves toward lower pressure.
- The second term magnetic force.
- The third term is the gravitational force.
- Other forces may be added based on the situation.

Approximations

- Density conservation

$$\frac{dn}{dt} = S - L \qquad \frac{\partial n}{\partial t} + u.\nabla n = S - L$$

- Momentum equation

$$m\frac{du}{dt} = \sum F \quad \Longrightarrow \quad \frac{du}{dt} = \sum F/m$$

$$\frac{\partial u}{\partial t} + u.\nabla u = \sum F/m \quad \Longrightarrow \quad \rho(\frac{\partial u}{\partial t} + u.\nabla u) = \rho \sum F/m$$

- The set of fluid equations is not closed
- An assumption for the equation of state and Maxwell equations are mandatory to close the equations set.

Thank You!