anode

1 cm

4.6 Ion Waves

In the absence of collisions, ordinary sound waves would not occur. Ions can still transmit vibrations to each other because of their charge, however; and acoustic waves can occur through the intermediary of an electric field. Since the motion of massive ions will be involved, these will be low-frequency oscillations, and we can use the plasma approximation of Sect. 3.6. We therefore assume $n_i = n_e = n$ and do not use Poisson's equation. The ion fluid equation in the absence of a magnetic field is

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We have assumed $\mathbf{E} = -\nabla\phi$ and used the equation of state. Linearizing and assuming plane waves, we have

$$-i\omega Mn_0 v_{i1} = -en_0 ik\phi_1 - \gamma_i KT_i ikn_1 \qquad (4.38)$$

As for the electrons, we may assume $m = 0$ and apply the argument of Sect. 3.5, regarding motions along $\mathbf{B}$, to the present case of $\mathbf{B} = 0$. The balance of forces on electrons, therefore, requires

$$n_e = n = n_0 \exp\left(\frac{e\phi_1}{KT_e}\right) = n_0\left(1 + \frac{e\phi_1}{KT_e} + \cdots\right)$$

The perturbation in density of electrons, and, therefore, of ions, is then

$$n_1 = n_0 \frac{e\phi_1}{KT_e} \qquad (4.39)$$

Here the n_0 of Boltzmann's relation also stands for the density in the equilibrium plasma, in which we can choose $\phi_0 = 0$ because we have assumed $\mathbf{E_0} = 0$. In linearizing Eq. (4.39), we have dropped the higher-order terms in the Taylor expansion of the exponential.
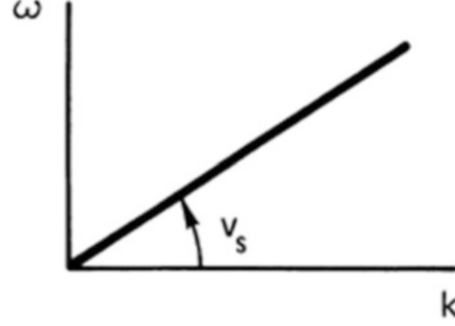
The only other equation needed is the linearized ion equation of continuity. From Eq. (4.22), we have

$$i\omega n_1 = n_0 ik v_{i1} \qquad (4.40)$$

In Eq. (4.38), we may substitute for ϕ_1 and n_1 in terms of v_{i1} from Eqs. (4.39) and (4.40) and obtain

$$i\omega Mn_0 v_{i1} = \left(en_0 ik\frac{KT_e}{en_0} + \gamma_i KT_i ik\right)\frac{n_0 ik v_{i1}}{i\omega}$$

$$\omega^2 = k^2\left(\frac{KT_e}{M} + \frac{\gamma_i KT_i}{M}\right)$$

Fig. 4.12 Dispersion
relation for ion acoustic
waves in the limit of small
Debye length

$$\boxed{\frac{\omega}{k} = \left(\frac{KT_e + \gamma_i KT_i}{M}\right)^{1/2} \equiv v_s} \tag{4.41}$$

This is the dispersion relation for *ion acoustic waves*; v_s is the sound speed in a plasma. Since the ions suffer one-dimensional compressions in the plane waves we have assumed, we may set $\gamma_i = 3$ here. The electrons move so fast relative to these waves that they have time to equalize their temperature everywhere; therefore, the electrons are isothermal, and $\gamma_e = 1$. Otherwise, a factor γ_e would appear in front of KT_e in Eq. (4.41).

The dispersion curve for ion waves (Fig. 4.12) has a fundamentally different character from that for electron waves (Fig. 4.5). Plasma oscillations are basically *constant-frequency waves*, with a correction due to thermal motions. Ion waves are basically *constant-velocity* waves and exist *only* when there are thermal motions. For ion waves, the group velocity is equal to the phase velocity. The reasons for this difference can be seen from the following description of the physical mechanisms involved. In electron plasma oscillations, the other species (namely, ions) remains essentially fixed. In ion acoustic waves, the other species (namely, electrons) is far from fixed; in fact, electrons are pulled along with the ions and tend to shield out electric fields arising from the bunching of ions. However, this shielding is not perfect because, as we saw in Sect. 1.4, potentials of the order of KT_e/e can leak out because of electron thermal motions. What happens is as follows. The ions form regions of compression and rarefaction, just as in an ordinary sound wave. The compressed regions tend to expand into the rarefactions, for two reasons. First, the ion thermal motions spread out the ions; this effect gives rise to the second term in the square root of Eq. (4.41). Second, the ion bunches are positively charged and tend to disperse because of the resulting electric field. This field is largely shielded out by electrons, and only a fraction, proportional to KT_e, is available to act on the ion bunches. This effect gives rise to the first term in the square root of Eq. (4.41). The ions overshoot because of their inertia, and the compressions and rarefactions are regenerated to form a wave.

The second effect mentioned above leads to a curious phenomenon. When KT_i goes to zero, ion waves still exist. This does not happen in a neutral gas (Eq. (4.36)). The acoustic velocity is then given by

$$v_s = (KT_e/M)^{1/2} \tag{4.42}$$

This is often observed in laboratory plasmas, in which the condition $T_i \ll T_e$ is a common occurrence. The sound speed v_s depends on *electron* temperature (because the electric field is proportional to it) and on *ion* mass (because the fluid's inertia is proportional to it).

4.7   Validity of the Plasma Approximation

In deriving the velocity of ion waves, we used the neutrality condition $n_i = n_e$ while allowing **E** to be finite. To see what error was engendered in the process, we now allow n_i to differ from n_e and use the linearized Poisson equation:

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E}_1 = \varepsilon_0 k^2 \phi_1 = e(n_{i1} - n_{e1}) \tag{4.43}$$

The electron density is given by the linearized Boltzmann relation Eq. (4.39):

$$n_{e1} = \frac{e\phi_1}{KT_e} n_0 \tag{4.44}$$

Inserting this into Eq. (4.43), we have

$$\varepsilon_0 \phi_1 \left(k^2 + \frac{n_0 e^2}{\varepsilon_0 KT_e} \right) = e n_{i1} \tag{4.45}$$

$$\varepsilon_0 \phi_1 \left(k^2 \lambda_D^2 + 1 \right) = e n_{i1} \lambda_D^2$$

The ion density is given by the linearized ion continuity equation (4.40):

$$n_{i1} = \frac{k}{\omega} n_0 v_{i1} \tag{4.46}$$

Inserting Eqs. (4.45) and (4.46) into the ion equation of motion Eq. (4.38), we find

$$i\omega M n_0 v_{i1} = \left(\frac{e n_o i k}{\varepsilon_0} \frac{e \lambda_D^2}{1 + k^2 \lambda_D^2} + \gamma_i KT_i i k \right) \frac{k}{\omega} n_0 v_{i1}$$

$$\omega^2 = \frac{k^2}{M} \left(\frac{n_0 e^2 \varepsilon_0^{-1} \lambda_D^2}{1 + k^2 \lambda_D^2} + \gamma_i KT_i \right) \tag{4.47}$$

$$\frac{\omega}{k} = \left(\frac{KT_e}{M} \frac{1}{1 + k^2 \lambda_D^2} + \frac{\gamma_i KT_i}{M} \right)^{1/2} \tag{4.48}$$

This is the same as we obtained previously (Eq. (4.41)) except for the factor $1 + k^2 \lambda_{\mathrm{D}}^2$. Our assumption $n_i = n_e$ has given rise to an error of order $k^2 \lambda_{\mathrm{D}}^2 = (2\pi \lambda_{\mathrm{D}}/\lambda)^2$. Since λ_{D} is very small in most experiments, the plasma approximation is valid everywhere except in a thin layer, called a sheath (Chap. 8), a few λ_{D}'s in thickness, next to a wall.

4.8 Comparison of Ion and Electron Waves

If we consider these short-wavelength waves by taking $k^2 \lambda_{\mathrm{D}}^2 \gg 1$, Eq. (4.47) becomes

$$\omega^2 = k^2 \frac{n_0 e^2}{\varepsilon_0 M k^2} = \frac{n_0 e^2}{\varepsilon_0 M} \equiv \Omega_p^2 \tag{4.49}$$

We have, for simplicity, also taken the limit $T_i \to 0$. Here Ω_p is the ion plasma frequency. For high frequencies (short wavelengths) the ion acoustic wave turns into a constant-frequency wave. There is thus a complementary behavior between electron plasma waves and ion acoustic waves: the former are basically constant frequency, but become constant velocity at large k; the latter are basically constant velocity, but become constant frequency at large k. This comparison is shown graphically in Fig. 4.13.

Experimental verification of the existence of ion waves was first accomplished by Wong, Motley, and D'Angelo. Figure 4.14 shows their apparatus, which was again a Q-machine. (It is no accident that we have referred to Q-machines so often; careful experimental checks of plasma theory were possible only after schemes to make quiescent plasmas were discovered.) Waves were launched and detected by grids inserted into the plasma. Figure 4.15 shows oscilloscope traces of the transmitted and received signals. From the phase shift, one can find the phase velocity (same as group velocity in this case). These phase shifts are plotted as
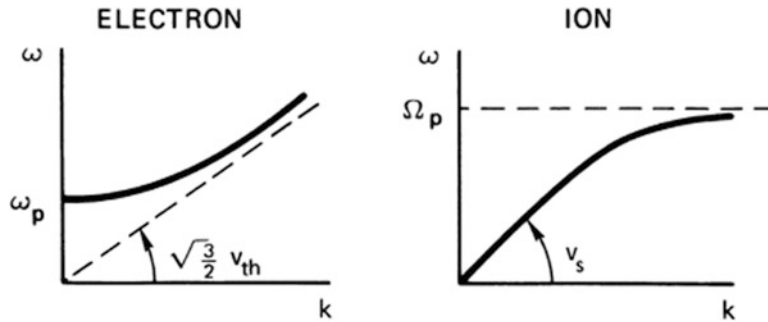

Fig. 4.13 Comparison of the dispersion curves for electron plasma waves and ion acoustic waves

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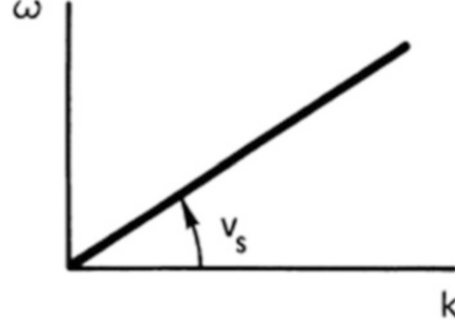
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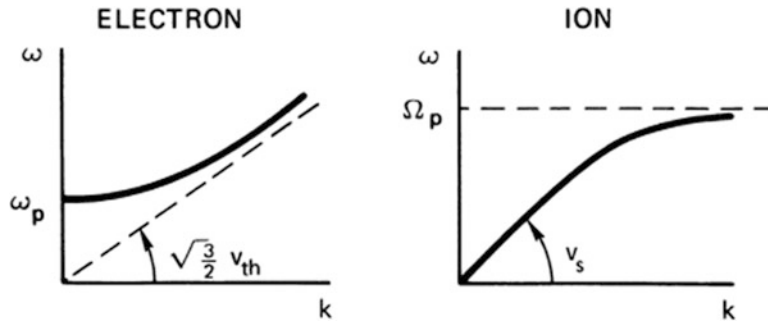

Fig. 4.13 Comparison of the dispersion curves for electron plasma waves and ion acoustic waves

①

$$m\,a = F$$

$$m\,\frac{d^2 x}{dt^2} = q E = -e E \qquad \nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot E = \frac{en}{\varepsilon_0}$$

$$m\,\frac{d^2 x}{dt^2} = -e\,\frac{en}{\varepsilon_0}\,x \qquad \frac{E}{x} = \frac{en}{\varepsilon_0}$$

$$\frac{d^2 x}{dt^2} = -\frac{e^2 n}{\varepsilon_0 m}\,x$$

$$\frac{d^2 x}{dt^2} + \left(\frac{e^2 n}{\varepsilon_0 m}\right) x = 0$$

Show that the equation of motion (Newton's Second law) for each electrons is given by: what is the physics behind this equation?

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x} n_e V_e = 0$$

$$m\left(\frac{\partial V_e}{\partial t} + V_e \frac{\partial V_e}{\partial x}\right) = -eE$$

$$\varepsilon_o \frac{\partial E}{\partial x} = \rho = e(n_i - n_e)$$

$$\frac{\partial E_1}{\partial x} = -\frac{e}{\varepsilon_o} n_{e1} \quad \text{—①}$$

$$m\frac{\partial V_{e1}}{\partial t} = -eE_1$$

$$\frac{\partial^2 V_{e1}}{\partial t \partial x} = -\frac{e}{m} \frac{\partial E_1}{\partial x} \quad \text{—②}$$

$$\frac{\partial^2 V_{e1}}{\partial t \partial x} = -\frac{e}{m} \ast -\frac{e}{\varepsilon_o} n_{e1}$$

$$\frac{\partial^2 V_{e1}}{\partial t \partial x} = \frac{e^2}{m\varepsilon_o} n_{e1} \quad \text{—③}$$

$$\frac{\partial n_{e1}}{\partial t} + n_{eo} \frac{\partial V_{e1}}{\partial x} = 0$$

$$\frac{\partial V_{e1}}{\partial x} = -\frac{1}{n_{eo}} \frac{\partial n_{e1}}{\partial t} \quad \text{—④ \quad in ③}$$

$$\frac{\partial n_{e1}}{\partial t^2} = -\left(\frac{e^2 n_{eo}}{m \varepsilon_o}\right) n_{e1}$$

Electron plasma waves

$$m_e \left(\frac{\partial V_e}{\partial t} + V_e \frac{\partial V_e}{\partial x} \right) = -e E - \frac{\partial P_e}{\partial x}$$

↓ n_e ↓ n_e

→ divided by n_e

$$m_e \left(\frac{\partial V_e}{\partial t} + V_e \frac{\partial V_e}{\partial x} \right) = -eE - \frac{1}{n_e} \frac{\partial P_e}{\partial x}$$

↓

$$P_e = 3 k_B T_e n_e$$

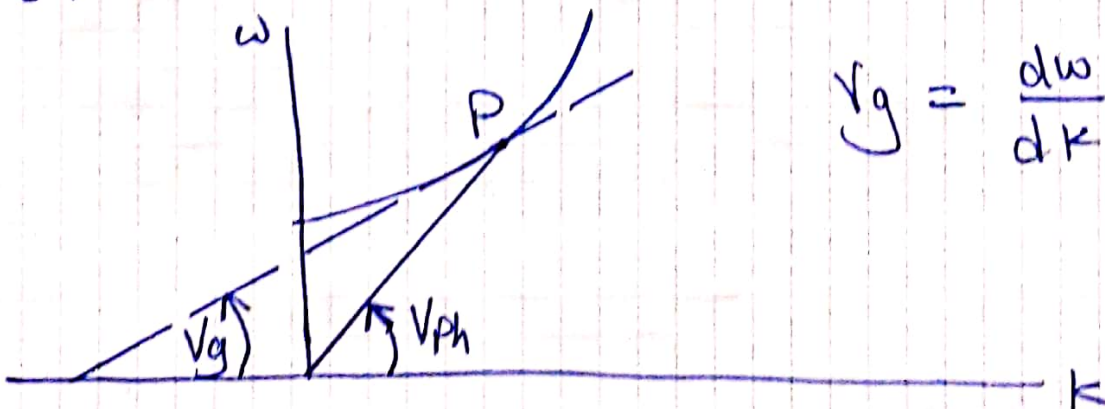$$\omega^2 = \omega_p^2 + \frac{3}{2} k^2 V_{th}^2$$

↓

$$\frac{2 k_B T_e}{m_e}$$

we use plan wave analysis

$$V_e = V_{e1} = V_{e1} \exp(ikx - i\omega t)$$

$$n_e = n_{e0} + n_{e1} \implies n_{e1} = n_1 \exp(ikx - i\omega t)$$

$$\frac{\partial}{\partial t} \longrightarrow -i\omega \qquad , \qquad \frac{\partial}{\partial x} \longrightarrow ik$$

$$V_g = \frac{d\omega}{dk} = \frac{3}{2} \frac{k}{\omega} V_{th}^2$$

Ion Waves

* ion time scale $\longrightarrow$ what about the electrons?

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x} n_i u_i = 0$$

$$m_i n_i \left(\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} \right) = e n_i E - \frac{\partial P_i}{\partial x}$$

$$E = - \frac{\partial \phi}{\partial x}$$

$$\frac{\partial E}{\partial x} = \frac{\rho}{\varepsilon_0} = \frac{e}{\varepsilon_0} (n_i - n_e) \qquad \text{Gauss}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\varepsilon_0} (n_e - n_i) \qquad \text{Poisson}$$

$$n_e = n_{eo} \exp \left(\frac{e \phi}{k_B T_e} \right)$$

Two cases

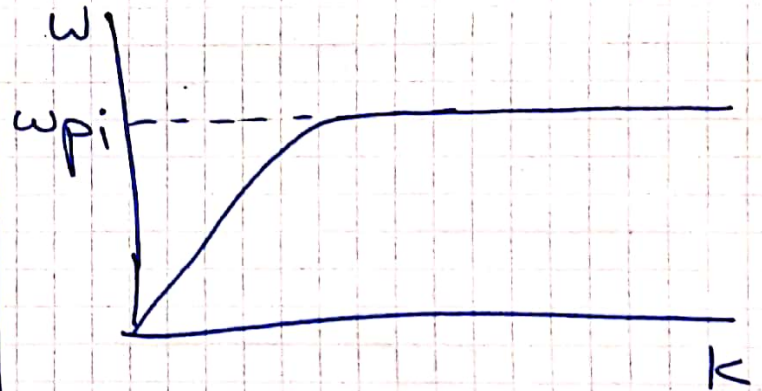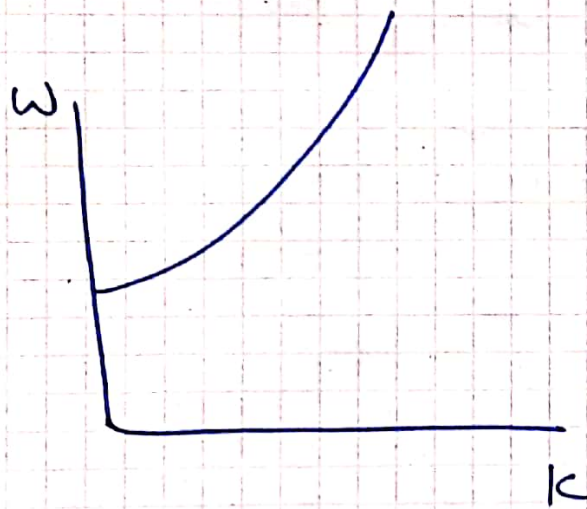|	
$n_e = n_i$	$\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\varepsilon_0} (n_e - n_i)$
neutrality cond.	Poisson eq
at ~~small~~ small wave number	
$\frac{\omega}{k} = \left(\frac{k_B T_e}{m_i} \right)^{1/2} = C_s$	$\frac{\omega}{k} = \left(\frac{k_B T_e}{m_i (1 + k^2 \lambda_D^2)} \right)^{1/2}$
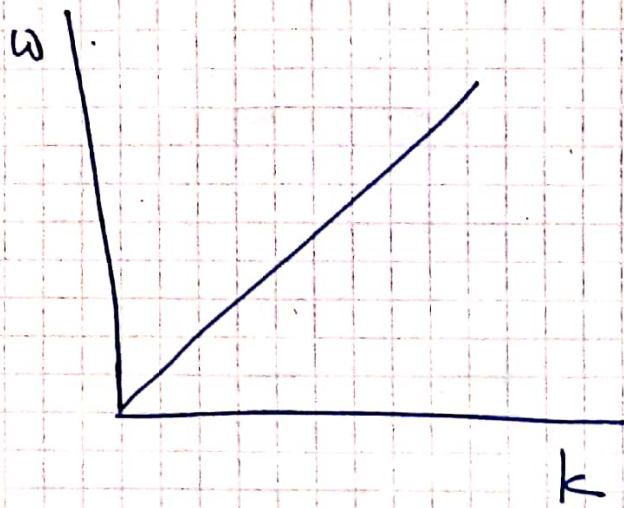
for $k^2 \lambda_D^2 \ll 1$

$$\frac{\omega}{k} = C_s$$

for $k^2 \lambda_D \gg 1$

$$\frac{\omega}{k} = \frac{C_s}{k \lambda_D} \Rightarrow \omega = \frac{C_s}{\lambda_D} = \omega_p$$

Comparison between e & i plasma waves

e	i
at $k=0 \Rightarrow \omega = \omega_{pe}$	at $k=0 \quad \omega = 0$
at $larg\ k \Rightarrow \frac{\omega}{k} \rightarrow const$	$\omega = \omega_{pi}$
∇P_e should be presented	∇P_i can be cancelled ∇P_e produce the prusser to wave formation