Waves & Dispersion

Ibrahem Elkamash, PhD

Physics Department, Faculty of Science, Mansoura University

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System response

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- Waves: $\omega_{Perturbation} \sim \omega_{Inertia}$

$$\text{System} + \text{Intermediate frequency perturbation} \rightarrow \text{Waves}$$

Beats (AM, FM)

-Suppose two coherent monochromatic waves:

$$\psi_1 = Acos(k_1 x - \omega_1 t), \qquad \psi_2 = Acos(k_2 x - \omega_2 t),$$

The superposition of two waves:

$$\begin{aligned}
\psi(x,t) &= \psi_1 + \psi_2, \\
&= 2Acos[\frac{(k_2 - k_1)}{2}x - \frac{(\omega_2 - \omega_1)}{2}t]cos[\frac{(k_2 + k_1)}{2}x - \frac{(\omega_2 + \omega_1)}{2}t], \\
&= 2Acos[\Delta k x - \Delta\omega t]cos[k_0 x - \omega_0 t], \\
&= A(x,t)cos[k_0 x - \omega_0 t],
\end{aligned}$$

where $\Delta k = \frac{(k_2 - k_1)}{2}$, $\Delta\omega = \frac{(\omega_2 - \omega_1)}{2}$, $k_0 = \frac{(k_2 + k_1)}{2}$, $\omega_0 = \frac{(\omega_2 + \omega_1)}{2}$.

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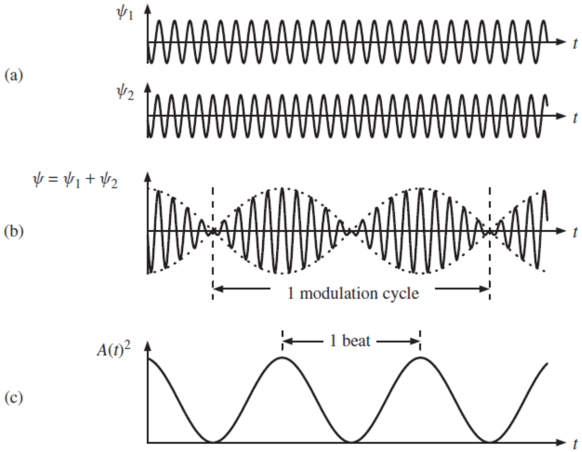

(a)
ψ_1
t
ψ_2
t
(b)
$\psi = \psi_1 + \psi_2$
t
1 modulation cycle
(c)
$A(t)^2$
1 beat
t

phase vs group velocity

- The velocity of a constant phase:

$$k_0 x - \omega_0 t = 0 \rightarrow \frac{dx}{dt} = \frac{\omega_0}{k_0} \doteq V_{ph},$$

-V_{ph}: the velocity of a constant phase.

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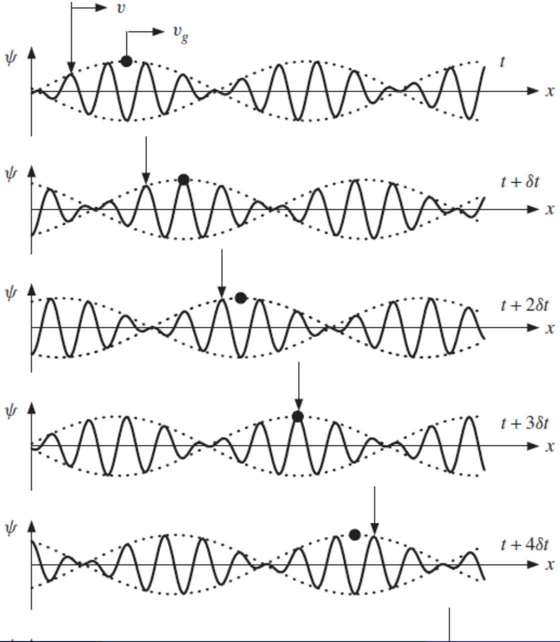
- The velocity of of constat amplitude $A(x, t)$:

$$\Delta kx - \Delta\omega t = 0 \to \frac{dx}{dt} = \frac{\Delta\omega}{\Delta k} \doteq V_g,$$

-V_g:the velocity of a constant amplitude.

phase vs group velocity



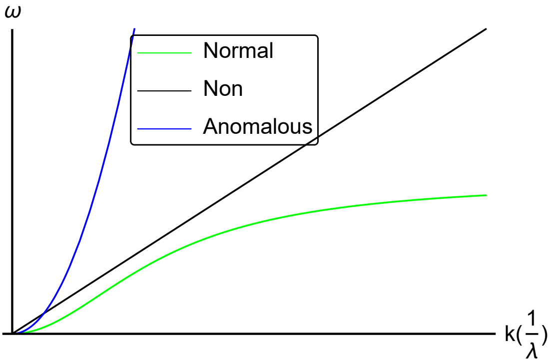
-The phase velocity $V_{ph} = \frac{\omega}{k}$.
-The group velocity $V_g = \frac{d\omega(k)}{dk}$.

$$\begin{aligned} V_g &= \frac{d\omega}{dk} \\ &= v_{ph} + k\frac{dV_{ph}}{dk} \\ &= V_{ph} - \lambda\frac{dV_{ph}}{d\lambda}, \end{aligned}$$

-Special Cases:

- Non dispersive: If $\frac{dV_{ph}}{d\lambda} = 0$, so $V_g = V_{ph}$: Vacuum
- Normal dispersive: If $\frac{dV_{ph}}{d\lambda} > 0$, so $V_g < V_{ph}$: Air
- Anomalous dispersive: If $\frac{dV_{ph}}{d\lambda} < 0$, so $V_g > V_{ph}$: Plasma

Medium types



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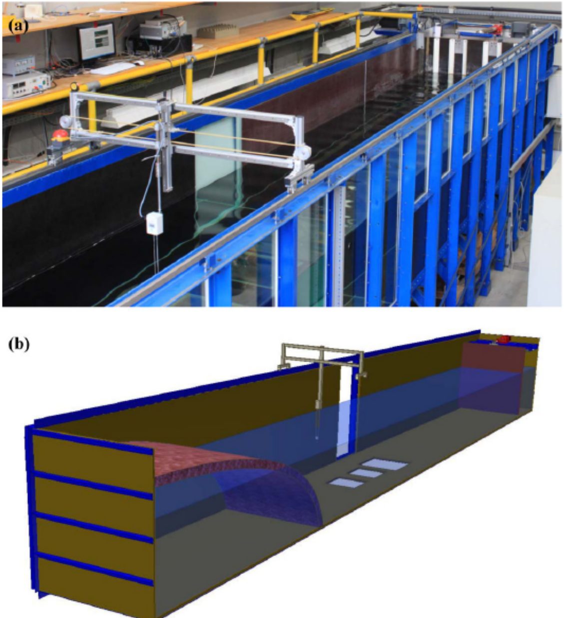
Water Waves



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-Cases:

- Shallow water ($kD \ll 1$) with no tension (S=0):
  $\tanh(kD) \approx kD \rightarrow \omega^2 = gDk^2$
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- Deep water ($kD \gg 1$) with no tension (S=0):
  $\tanh(kD) \approx 1 \rightarrow \omega^2 = gk$
  $V_{ph} = \sqrt{\frac{g}{k}} \propto \lambda$, $V_g = \frac{1}{2}\sqrt{\frac{g}{k}}$
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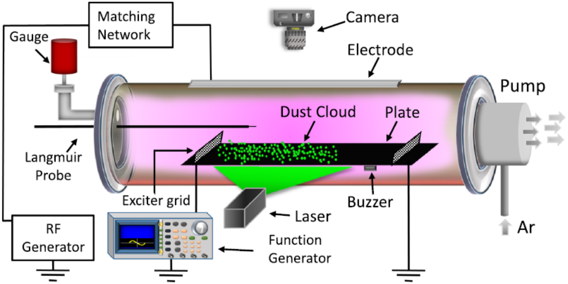
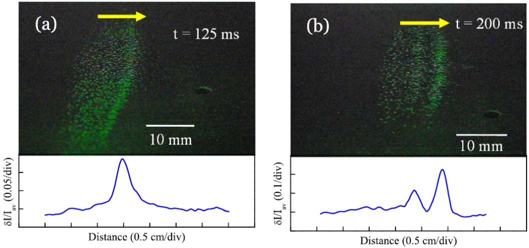
- shallow water ($kD \ll 1$) with surface tension ($S \neq 0$):
  $\tanh(kD) \approx kD \rightarrow \omega^2 = gDk^2 + \frac{SD}{\rho}k^4$
  $V_{ph} \approx \sqrt{\frac{SD}{\rho}}k \propto \frac{1}{\lambda},\ V_g \approx 2\sqrt{\frac{SD}{\rho}}k$
  $V_g = 2V_{ph} \rightarrow$ anomalous-dispersive.

Plasma Waves





Plasma waves

- Plasma shielding ($\omega \ll kV_{the} \ll kV_{thi}$)

$$k^2 = -\sum \frac{\omega_{ps}^2}{V_{ths}^2} = -\frac{1}{\lambda_D},$$

where λ_D is the overall plasma Debye shielding length.

- Plasma oscillations ($\omega \gg kV_{the} \gg kV_{thi}$):

$$\omega^2 = \omega_{pi}^2 + \omega_{pe}^2,$$

- Plasma waves (Ion plasma wave) ($kV_{thi} \lesssim \omega \ll kV_{the}$):

$$\omega^2 = V_{thi}^2 k^2 + \frac{c_i^2 k^2}{1 + \lambda_{De}^2 k^2},$$

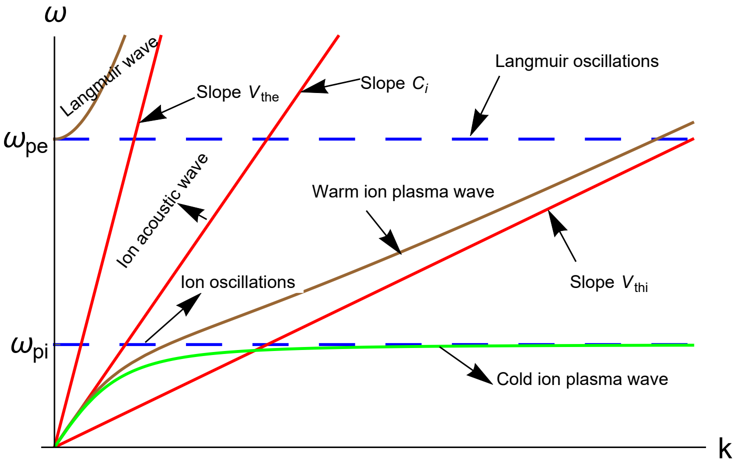
- Plasma waves (Electron plasma wave) ($kV_{thi} \ll \omega \lesssim kV_{the}$):

$$\omega^2 = \omega_{pe}^2 + V_{the}^2 k^2.$$

- Plasma waves (Electron and ion plasma wave) ($kV_{thi}, kV_{the} \lesssim \omega$):

$$1 = \frac{\omega_{pi}^2}{\omega^2 - k^2 V_{thi}^2} + \frac{\omega_{pe}^2}{\omega^2 - k^2 V_{the}^2},$$

Plasma waves



Plasma Waves Terminology

- **Propagation direction** (**k**):
  - Longitudinal Waves: $\mathbf{E}//\mathbf{k}$.
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- **Perturbed magnetic field** ($\mathbf{B}$):
  - Electrostatic Waves: $\mathbf{E} // \mathbf{k} \rightarrow \mathbf{B} = 0$
  - Electromagnetic Waves: $\mathbf{E} \perp \mathbf{k} \rightarrow \mathbf{B} \neq 0$

Thanks for your attention!