Plasma Models

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Introduction

- **Defination**: a special class of gases made up of a large number of electrons and ionized atoms and molecules, in addition to neutral atoms and molecules as are present in a normal (non-ionized) gas.
- **Aim**: Studing the dynamics (Knowing the position and velocity at instant time t) of the plasma
- **Models**: Depending on the density of charged particles, a plasma behaves either as a fluid, with collective effects being dominant, or as a collection of individual particles.

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I- Single-particle model.

II- Kinetic model.

III- Fluid model.

Single Particle model (Liouville Eqs) #1

- The plasma is a collection of charged particles. So in order to study various physical phenomena inside the plasma, we have to solve the equations of motion:

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \tag{1}$$

$$m_i \frac{d\mathbf{v}_i}{dt} = \mathbf{F}, \tag{2}$$

  for each particle.
- Where the position vector **r** is given by

$$\mathbf{r} = x\mathbf{x} + y\mathbf{y} + z\mathbf{z}. \tag{3}$$

  and the velcoity vector **v** is given by

$$\mathbf{v} = v_x\mathbf{x} + v_y\mathbf{y} + v_z\mathbf{z}. \tag{4}$$

- **F** is the combined influence forced, due to the externally applied forces and the internal forces generated by all the other plasma particles

Example (for single particle $i = 1$): $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$.

- With only a magnetic field present ($\mathbf{E} = 0, \mathbf{B} = B_0\mathbf{z}$): The movement of charged particles is restricted to circular motion known as **gyration** in a direction perpendicular to the magnetic field plus uninhibited motion along the magnetic field.

Single Particle model #2

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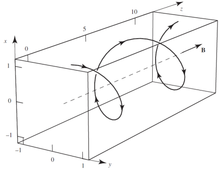

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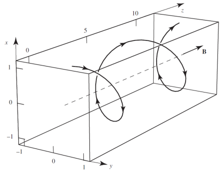
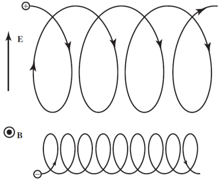
- With only a magnetic field present ($\mathbf{E} = 0, \mathbf{B} = B_0\mathbf{z}$): The movement of charged particles is restricted to circular motion known as **gyration** in a direction perpendicular to the magnetic field plus uninhibited motion along the magnetic field.
- The addition of a static electric field ($\mathbf{E} = E_0\mathbf{x}, \mathbf{B} = B_0\mathbf{z}$): Particles with both positive and negative charges to **drift** in a direction perpendicular to both the magnetic and the electric fields.





Comments:

- If the plasma consists of **N** particles, we need to solve **6N** coupled nonlinear differential equation simultaneously.
- Hence, it will be an impossible task to solve this problem analytically and it will be waste of time and money computationally.
- A plasma is a system containing a very large number of interacting charged particles, so that for its analysis it is appropriate and convenient to use **a statistical approach** to describe the positions and velocities of plasma particles using **a probability distribution function**.
- Describing a plasma using a distribution function is known as **plasma kinetic theory**.

Kinetic model #1

- **Configuration space r**: the location of each particle is documented by a position vector **r** drawn from the origin to the physical point at which the particle resides. In other words, we have

$$\mathbf{r} = x\mathbf{x} + y\mathbf{y} + z\mathbf{z}. \tag{5}$$

  We consider a small elemental volume $d\mathbf{r} = dxdydz$, also denoted as d^3r.

- **Velocity space v**: the location of the particle in this velocity space and it is given by:

$$\mathbf{v} = v_x\mathbf{x} + v_y\mathbf{y} + v_z\mathbf{z}. \tag{6}$$

  In analogy with configuration space, we think of the components $v_x, v_y, and v_z$ as being coordinates in *velocity space*.

- **Phase space**: defined by the six coordinates x, y, z, v_x, v_y, and v_z. Thus, the position $\mathbf{r}$ and the velocity $\mathbf{v}$ of a particle at any given time can be represented as a point in this six-dimensional space.

$$\mathbf{d}V = d\mathbf{r}d\mathbf{v} = d^3rd^3v = dxdydzdv_xdv_ydv_z \tag{7}$$

- **Velocity distribution function** $f_s(t, \mathbf{r}, \mathbf{v})$:

$$f_s(t, \mathbf{r}, \mathbf{v})dV = f_s(t, \mathbf{r}, \mathbf{v})d^3rd^3v = dN \tag{8}$$

the number of particles in a volume element dV in phase space at time t.

- $f_s(t, \mathbf{r}, \mathbf{v})$: the no of particles per unit volume in phase space at time t.
- $\int_{-\infty}^{\infty} f_s(t, \mathbf{r}, \mathbf{v})d\mathbf{v} = n(\mathbf{r}, t)$: the number particle density in real space only at time t.

Kinetic model #3

- $\int_{-\infty}^{\infty} \hat{f}_s(t, \mathbf{r}, \mathbf{v}) d\mathbf{r} d\mathbf{v} = N$: the total number density.

- $\hat{f}_s(t, \mathbf{r}, \mathbf{v}) = \frac{f_s(t,\mathbf{r},\mathbf{v})}{N} = P_s(t, \mathbf{r}, \mathbf{v})$: the normalized distribution function.

- $\int_{-\infty}^{\infty} P_s(t, \mathbf{r}, \mathbf{v}) d\mathbf{r} d\mathbf{v} = 1$

- $\hat{f}_s(t, \mathbf{r}, \mathbf{v}) d\mathbf{r} d\mathbf{v} = dP(t)$: the probability of finding a particle in volume element dV at time t.

- $\hat{f}_s(t, \mathbf{r}, \mathbf{v}) = P_s(t, \mathbf{r}, \mathbf{v})$: probability per unit volume of phase space is the **probability density**.

Kinetic model #3

- The evolution of the distribution function f_s in six-dimensional phase space (3 space + 3 velocity coordinates) can be described by

$$\frac{df_s(t, \mathbf{r}, \mathbf{v})}{dt} = \left(\frac{\partial f_s}{\partial t}\right)_{\mathrm{coll}}, \tag{9}$$

which is **a plasma kinetic equation**. Equation (9) can be understood as a continuity equation in the phase space, where

I- If $\left(\frac{\partial f_s}{\partial t}\right)_{\mathrm{coll}} > 0$: Ionization.

II- If $\left(\frac{\partial f_s}{\partial t}\right)_{\mathrm{coll}} < 0$: Recombination. III- If $\left(\frac{\partial f_s}{\partial t}\right)_{\mathrm{coll}} < 0$:

attachment.

- **Convective derivative**:

$$\frac{d\mathbf{A}(t,x)}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \frac{dx}{dt}\frac{\partial \mathbf{A}}{\partial x} = \frac{\partial \mathbf{A}}{\partial t} + v_x \frac{\partial \mathbf{A}}{\partial x} \tag{10}$$

- Generalizing the expression to three dimensions, we can express the convective derivative as

$$\frac{d\mathbf{A}(t,\mathbf{r})}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A} \tag{11}$$

- So, the total derivative of the distribution function in phase space can be written as

$$\frac{df_s(t,\mathbf{r},\mathbf{v})}{dt} = \left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}} + \frac{\mathbf{F}}{m_s}\cdot\nabla_{\mathbf{v}}\right] f_s(t,\mathbf{r},\mathbf{v}) \tag{12}$$

- where

$$\nabla_{\mathbf{r}} = \frac{\partial}{\partial x}\mathbf{x} + \frac{\partial}{\partial y}\mathbf{y} + \frac{\partial}{\partial z}\mathbf{z}, \quad (13)$$

$$\nabla_{\mathbf{v}} = \frac{\partial}{\partial v_x}\mathbf{x} + \frac{\partial}{\partial v_y}\mathbf{y} + \frac{\partial}{\partial v_z}\mathbf{z}, \quad (14)$$

are the gradient operator in three-dimensional configuration and velocity coordinates.

- So that the second and third terms in Eq. (9) are:

$$\left[\mathbf{v} \cdot \nabla_{\mathbf{r}}\right] f_s(t, \mathbf{r}, \mathbf{v}) = v_x \frac{\partial f_s}{\partial x}\mathbf{x} + v_y \frac{\partial f_s}{\partial y}\mathbf{y} + v_z \frac{\partial f_s}{\partial z}\mathbf{z}, \quad (15)$$

$$\left[\frac{\mathbf{F}}{m_s} \cdot \nabla_{\mathbf{v}}\right] f_s(t, \mathbf{r}, \mathbf{v}) = \frac{F_x}{m_s}\frac{\partial f_s}{\partial v_x}\mathbf{x} + \frac{F_y}{m_s}\frac{\partial f_s}{\partial v_y}\mathbf{y} + \frac{F_z}{m_s}\frac{\partial f_s}{\partial v_z}\mathbf{z}. \quad (16)$$

- So,the three-dimensional plasma kinetic equation becomes:

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}} + \frac{\mathbf{F}}{m_s}\cdot\nabla_{\mathbf{v}}\right] f_s(t,\mathbf{r},\mathbf{v}) = \left(\frac{\partial f_s}{\partial t}\right)_{\text{coll}} \tag{17}$$

- **Special cases**:

I- If $\left(\frac{\partial f_s}{\partial t}\right)_{\text{coll}} = C(f_s)$: It is called '**Boltzmann**' equation, where $C(f_s)$ is the Coloumb collision operator.

II- If $\left(\frac{\partial f_s}{\partial t}\right)_{\text{coll}} = FP(f_s)$: It is called '**Fokker-Plank**' equation, where $FP(f_s)$ is the FP collision operator.

III- If $\left(\frac{\partial f_s}{\partial t}\right)_{\text{coll}} = 0$: It is called '**Vlasov**' equation. Thus the '**Vlasov**' equation (9) can be simply stated as

$$\frac{df_s}{dt} = 0, \tag{18}$$

Comments:

- The **measurable or macroscopic** (i.e., ensemble average) values of various plasma parameters (e.g., density, flux, current) can be can easily be derived from the moments of distribution function $f_s(t, \mathbf{r}, \mathbf{v})$.
- **For example:** The total number $N(t, \mathbf{r})d\mathbf{r}$ of velocity points in the entire velocity space, is given by

$$N(t, \mathbf{r}) = \int_{-\infty}^{\infty} f(t, \mathbf{r}, \mathbf{v}) d\mathbf{v} = \int\int\int_{-\infty}^{\infty} f(t, \mathbf{r}, \mathbf{v}) dv_x dv_y dv_z. \tag{19}$$

- Consider any property $g(\mathbf{r}, \mathbf{v}, t)$ of a particle. The value of this quantity averaged over all velocities (**weighted average**)is then given by

$$\bar{g}_{av}(t, \mathbf{r}) =< g(t, \mathbf{r}, \mathbf{v}) >= \int_{-\infty}^{\infty} g(t, \mathbf{r}, \mathbf{v}) \hat{f}(t, \mathbf{r}, \mathbf{v}) d\mathbf{v}. \tag{20}$$

Fluid model #1

- Under certain assumptions it is not necessary to obtain the actual distribution function if one is only interested in the macroscopic values. Instead of first solving the Boltzmann (or Vlasov) equation for the distribution function and then integrating, it is possible to first take appropriate integrals over the Boltzmann equation and then solve for the quantities of interest.
- This approach is referred to as "**taking the moments of the Boltzmann equation.**" The resulting equations are known as the macroscopic transport equations, and form the foundation of **plasma fluid theory.**
- The basic procedure for deriving macroscopic equations from the Boltzmann equation involves multiplying it by powers of the velocity vector v and integrating over velocity space. It is important to realize that in performing such an integration we intrinsically lose information on the details of the velocity distribution.

Fluid model #2

- **The zeroth-order moment: continuity equation**
  To evaluate the zeroth-order moment, we multiply Eq. (9) by $v_0 = 1$ and integrate to find

$$\int \frac{\partial f_s}{\partial t} d\mathbf{v} + \int (\mathbf{v} \cdot \nabla_{\mathbf{r}}) f_s d\mathbf{v} + \frac{q_s}{m_s} \int [(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}}] f_s d\mathbf{v} = \int \left(\frac{\partial f_s}{\partial t} \right. \quad (21)$$

- **The continuity equation for mass or charge transport**:

Particle conservation $$\frac{\partial N_s(t, \mathbf{r})}{\partial t} + \nabla \cdot [N_s(t, \mathbf{r}) \mathbf{u}_s(t, \mathbf{r})] = 0. \quad (22)$$

- In the presence of collisions, a more general version of the continuity

$$\frac{\partial N_s}{\partial t} + \nabla \cdot [N_s \mathbf{u}_s] = -\alpha N_s^2 - \nu_a N_s + \nu_i N_s. \quad (23)$$

where α is the recombination rate, ν_a is the attachment rate, and ν_i is the ionization rate.

Fluid model #3

- **The first-order moment: momentum transport equation**
  The first-order moment of the Boltzmann equation is obtained by multiplying Eq. (9) by $m\mathbf{v}$ and integrating to find
- **The momentum transport equation (force balance)**:

$$m_s N_s \left[\frac{\partial}{\partial t} + \mathbf{u}_s \cdot \nabla\right]\mathbf{u}_s = q_s N_s \Big(\mathbf{E} + \mathbf{u}_s \times \mathbf{B}\Big) - \nabla \cdot \boldsymbol{\Psi}_s + \nabla \cdot \boldsymbol{\Pi}_s + \mathbf{R}_{ij}, \qquad (24)$$

The first term in right hand side (R.H.S) represents the Lorentz force density, while the second term $\nabla \cdot \boldsymbol{\Psi}_s$ is the pressure tensor force density, third term $\nabla.\boldsymbol{\Pi}$ is the viscous force density and the last term $\mathbf{R}_{ij}$ is the frictional force due to Coulomb collisions between species.

- **The second-order moment: energy transport equation** The second-order moment of the Boltzmann equation, i.e., the equation of energy conservation, is obtained by multiplying Eq. (9) by $1/2mv^2$ and integrating over velocity space,
- **The energy-conservation equation** can be written as:

$$\frac{\partial \frac{3}{2} P_s}{\partial t} + \nabla \cdot (\frac{3}{2} P \mathbf{u}_s) = P_s \nabla \cdot \mathbf{u}_s + \nabla \cdot q_s + \mathbf{R}_{ij}, \qquad (25)$$

- The quantity $\frac{3}{2}P_s$ represents the flow of energy density at the fluid velocity, or the macroscopic energy flux.
- The first term in R.H.S, $P_s \nabla \cdot \mathbf{u}_s$ represents the heating or cooling of the fluid due to compression or expansion of its volume.
- The new quantity $\mathbf{q}_s$ is the heat-flow (or heat-flux) vector, which represents microscopic energy flux.

<u>Comments</u>:

- We **could** in principle proceed by evaluating higher and higherorder moments of the Boltzmann equation.
- **However**, the equations of conservation of particle number, momentum, and energy are useful in making general statements about plasmas, but they cannot be considered as a closed system of plasma equations.
- **But**, in calculating each moment of the Boltzmann equation, however, we always obtained an equation that contained the next moment. In the zeroth-order moment the change in particle density was expressed as a function of the mean fluid velocity. In the first-order moment, the change in mean fluid velocity was expressed as a function of the pressure tensor. The second-order moment is an expression for the change in the pressure tensor, but brings in a new heat-flow term.
- **Every time** we obtain a new equation a new unknown appears, so that the number of equations is never sufficient for the determination

Fluid model #6

- **Complete set of Multiple-fluid equations**:

$$\frac{\partial N_s}{\partial t} + \nabla \cdot [N_s \mathbf{u}_s] = 0, \tag{26}$$

$$m_s N_s \left[\frac{\partial}{\partial t} + \mathbf{u}_s \cdot \nabla \right] \mathbf{u}_s = q_s N_s \left(\mathbf{E} + \mathbf{u}_s \times \mathbf{B} \right) - \nabla P_s + \nabla \cdot \mathbf{\Pi}_s + \mathbf{R}_{ij} \tag{27}$$

$$\text{Gauss' Law} \quad \nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0}, \tag{28}$$

$$\text{Gauss' Law} \quad \nabla \cdot \mathbf{B} = 0, \tag{29}$$

$$\text{Faraday's Law} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{30}$$

$$\text{Ampére's Law} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{31}$$

where

$$\text{The charge density} \quad \rho_q = \sum q_s N_s, \tag{32}$$

$$\text{The current density} \quad \mathbf{J} = \sum q_s N_s \mathbf{u}_s \tag{33}$$

Fluid model #7

- As we can see from Maxwell's equations that the plasma is coupled with the electromagnetic field in Maxwell's equation through ρ_q and $\mathbf{J}$.
- The plasma pressure density P can be determined by using the equation of state which is a relation between the pressure, plasma density and plasma temperature. For example:
  - **Cold state**: $P = 0$.
  - **Isothermal state**: $P_s = K_B N_s T_s$ holds for relatively slow time variations, where the plasma fluid can exchange energy with its surroundings allowing temperatures to reach equilibrium.
  - **Adiabatic state**: $P_s = C_s N_s^{\gamma}$.
- The frictional forces $\mathbf{R}_{ij} = -\sum_j m N_j \nu_{ij} (\mathbf{u}_i - \mathbf{u}_j)$.
- Also, the viscous force density $\nabla . \mathbf{\Pi} = m_s N_s \eta \nabla^2 \mathbf{u}_s$ where η is the kinematic viscosity coefficient.

Validity of the fluid model.:

- The fluid model describes a weakly coupled plasma system.
- This means that the average binding energy must be very small compared to the thermal energy.
- The classical coupling parameter represents the ratio of the average Coulomb potential energy to the average kinetic or thermal energy, i.e. $\Gamma_C = \frac{E_C}{E_{th}}$, where E_C is the Coulomb potential energy and E_{th} is the thermal energy. Consequently, the coupling parameter must be much less than 1, i.e. $\Gamma_C \ll 1$, to apply a fluid model to describe plasma problems.

Further reading

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Thanks for your attention!