Plasama Instability

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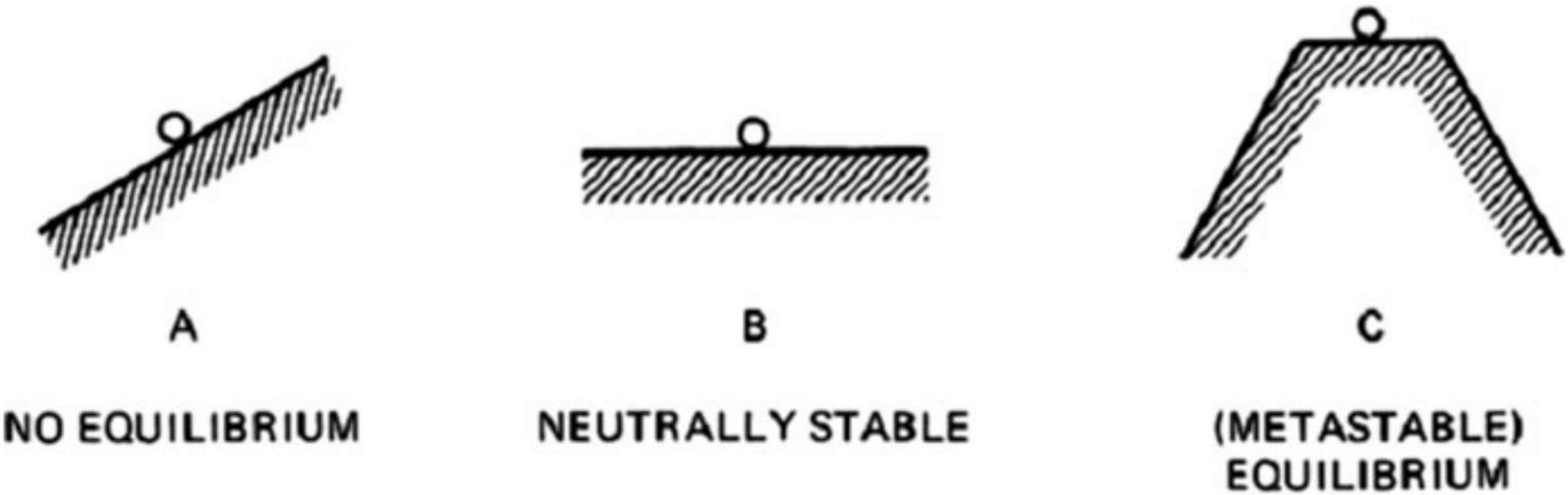
Opening Remarks

- Not everything I said is included in the slides.
- Please pay attention to the lectures and catch the most essential points.
- Form your own opinions and thoughts.
- Should ask questions.
- During the meeting let us exchange ideas about the lecture.

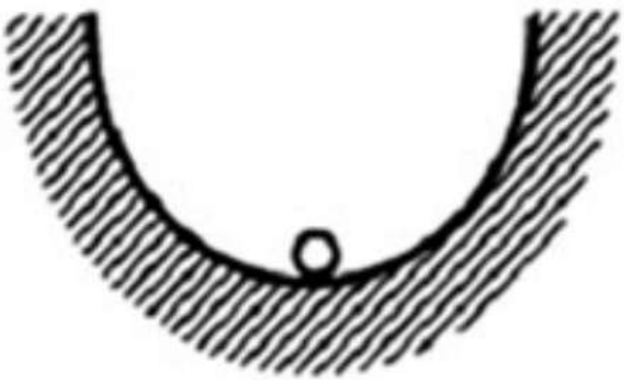
Outline

- **Difference Between Equilibrium & Stability**
- **Concept of β**
- **Classification of Instabilities**
- **Two Stream Instability**
- **Tutorial with Mathematica**

Equilibrium & Stability



Equilibrium & Stability, cont.



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STABLE EQUILIBRIUM

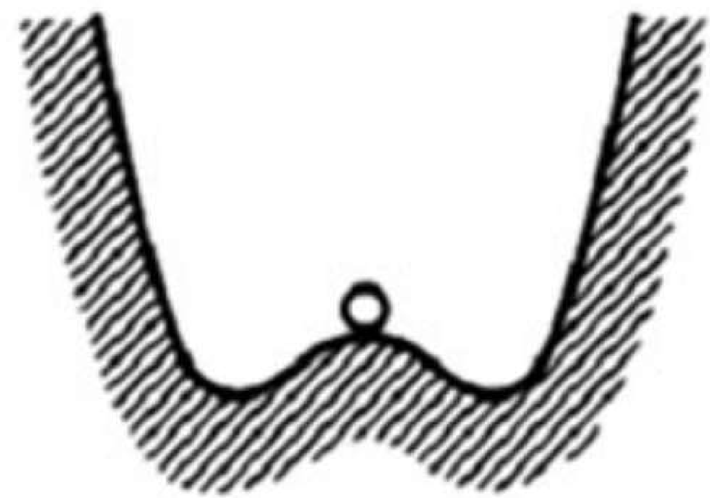
E

UNSTABLE EQUILIBRIUM

Equilibrium & Stability, cont.

F

EQUILIBRIUM WITH LINEAR STABILITY AND NONLINEAR INSTABILITY



G

EQUILIBRIUM WITH LINEAR INSTABILITY AND NONLINEAR STABILITY

Concept of β

- **Let us go to the whiteboard**

Concept of β

From Newton's Law

$$m \frac{dV}{dt} = F \quad \text{for one particle}$$

$$mn \frac{dV}{dt} = nF \quad \text{for } n \text{ particles}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V \frac{\partial}{\partial x}$$

For ions

$$Mn\left[\frac{\partial V_i}{\partial t} + V_i \frac{\partial V_i}{\partial x}\right] = en(E + V_i \times B) - \nabla P_i + Mng \quad (1)$$

For electrons

$$mn\left[\frac{\partial V_e}{\partial t} + V_e \frac{\partial V_e}{\partial x}\right] = -en(E + V_e \times B) - \nabla P_e + mng \quad (2)$$

Add 1 + 2

$$n\frac{\partial}{\partial t}(MV_i + mV_e) = en(V_i - V_e)\times B - \nabla(P_i + P_e) + n(M+m)g \rightarrow (3)$$

Let $P = P_i + P_e$, $n_i \approx n_e = n$

mass density $\rho = n_i M + n_e m \approx n(M+m)$

$$n_i M V_i + n_e m V_e = n(MV_i + mV_e)$$

let that $V_i \sim V_e = V$

$$\approx nV(M+m) = \rho V$$

$\Leftrightarrow$ 1.2 + 1.3 = 2.5 $\approx$ 2.4

1.3 − 1.2 = 0.1

$\Leftrightarrow$

Eq. (3)

$$\rho\frac{\partial V}{\partial t} = j\times B - \nabla P + \rho g \rightarrow (4)$$

At steady state $\left(\frac{\partial}{\partial t} = 0\right)$ and $g = 0$

$$\bar{\nabla} P = \bar{j} \times \bar{B}$$

But $\bar{\nabla} \times \bar{B} = \mu_0 \bar{j}$

$$\bar{\nabla} P = \frac{1}{\mu_0} (\bar{\nabla} \times \bar{B}) \times \bar{B}$$

$$= \frac{1}{\mu_0} \left[(\bar{B} \cdot \bar{\nabla}) \bar{B} - \frac{1}{2} \bar{\nabla} B^2 \right]$$

$$\nabla \left(P + \frac{B^2}{2\mu_0} \right) = \frac{1}{\mu_0} (B \cdot \nabla) B$$

$(B \cdot \nabla) B$ → = Zero

$$\therefore P + \frac{B^2}{2\mu_0} = \text{const.}$$

P → thermal pressure

$\frac{B^2}{2\mu_0}$ → magnetic field pressure

$$\beta = \frac{\text{Particle Pressure}}{\text{Mag. field Pressure}}$$

$$\beta = \frac{\sum n k_B T}{B^2 / 2\mu_0}$$

→ high → space

→ low → fusion (< 0.1)

Types of Instabilities

- Streaming instabilities
- Rayleigh–Taylor instabilities
- Kelvin-Helmholtz instability
- Universal instabilities
- Kinetic instabilities
- Gravitational Instability
- Resistive Drift Waves
- Weibel Instability

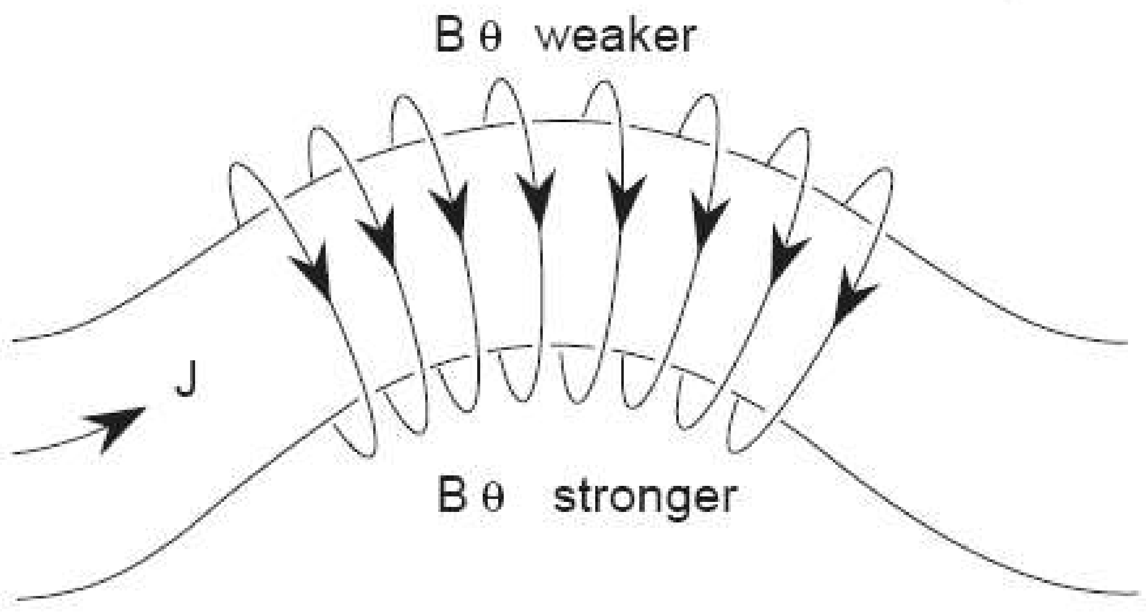
Types of Instabilities, cont.

Tokamak Instabilities

- **Kink instabilities**
- **Tearing mode**
- **Sawteeth**
- **Ballooning**
- **Fishbones**
- **Resistive wall mode**
- **Microinstabilities e.g. due to Ion/Electron Temperature Gradient driven mode OR Trapped Electron Mode**

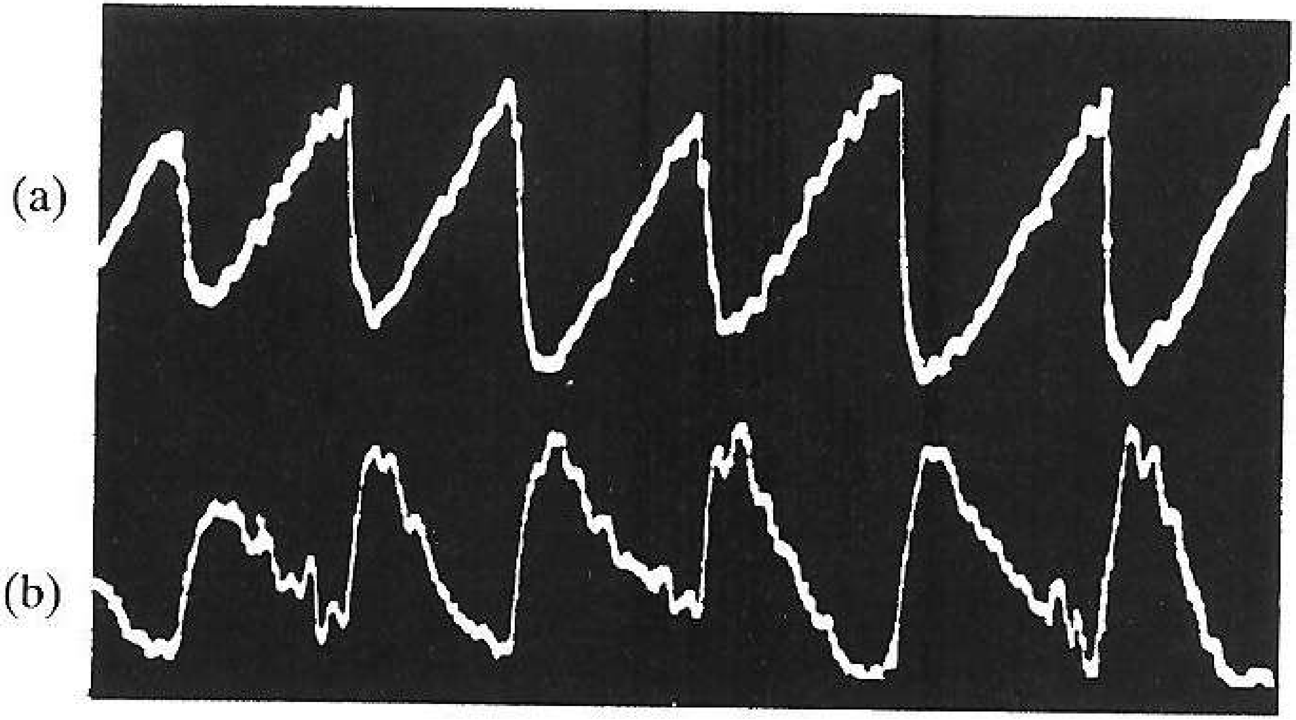
Types of Instabilities, cont.

- **Kink instability**



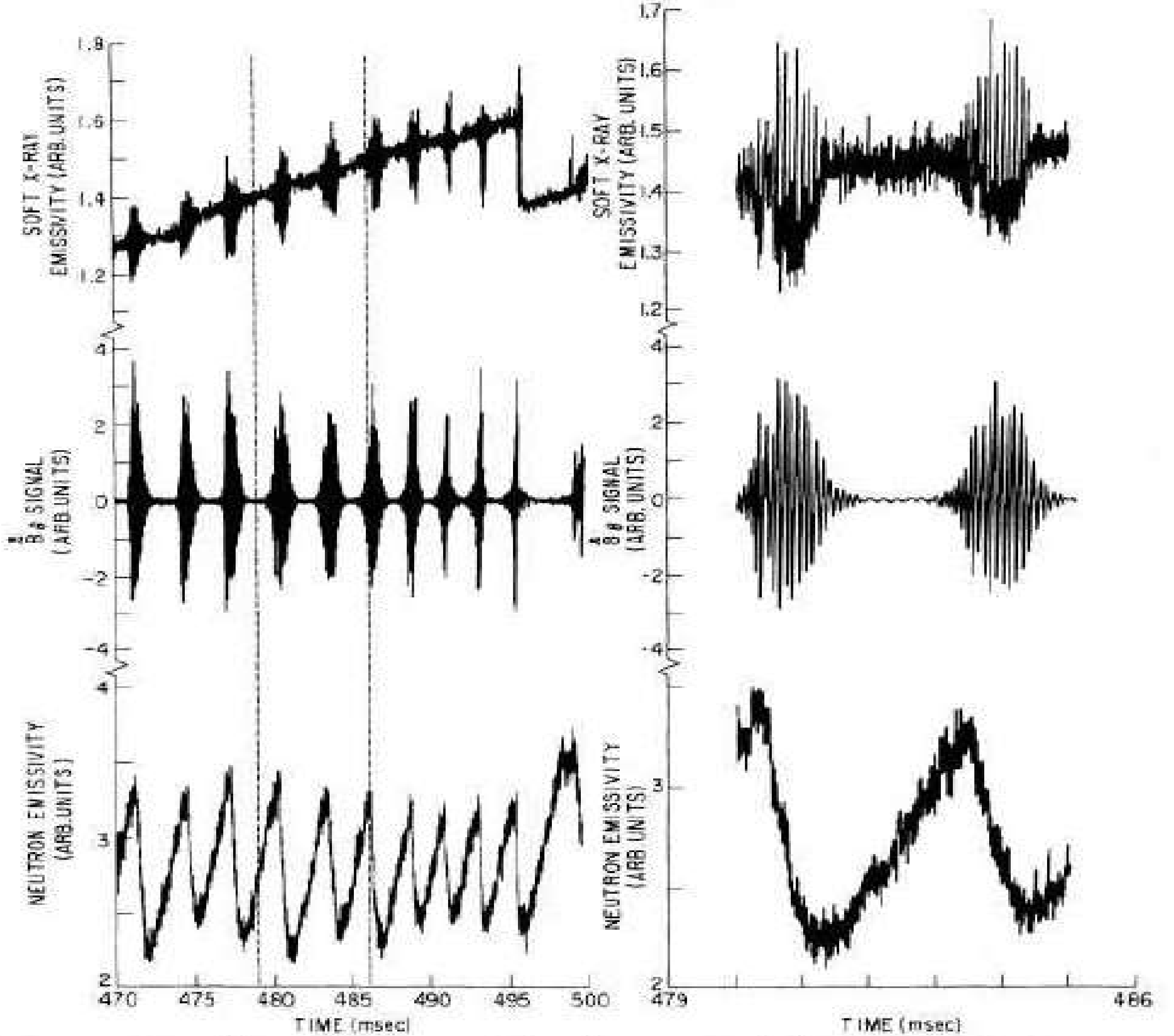
Types of Instabilities, cont.

- **Sawteeth**



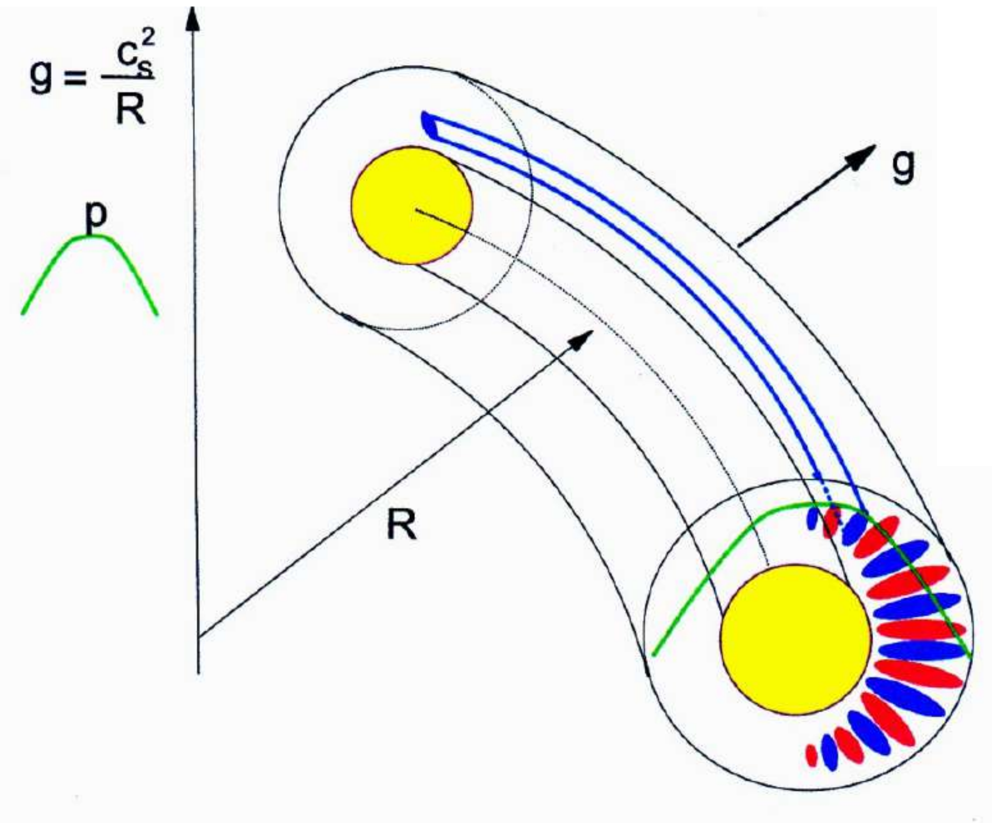
Types of Instabilities, cont.

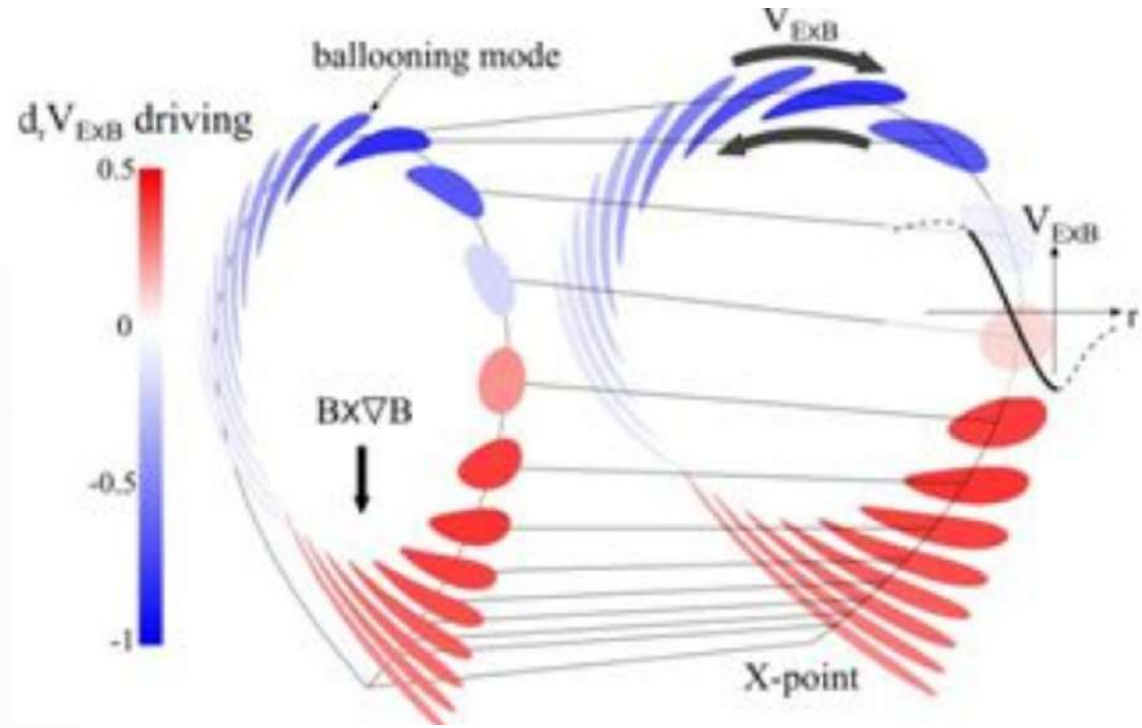
- **Fishbone**



Types of Instabilities, cont.

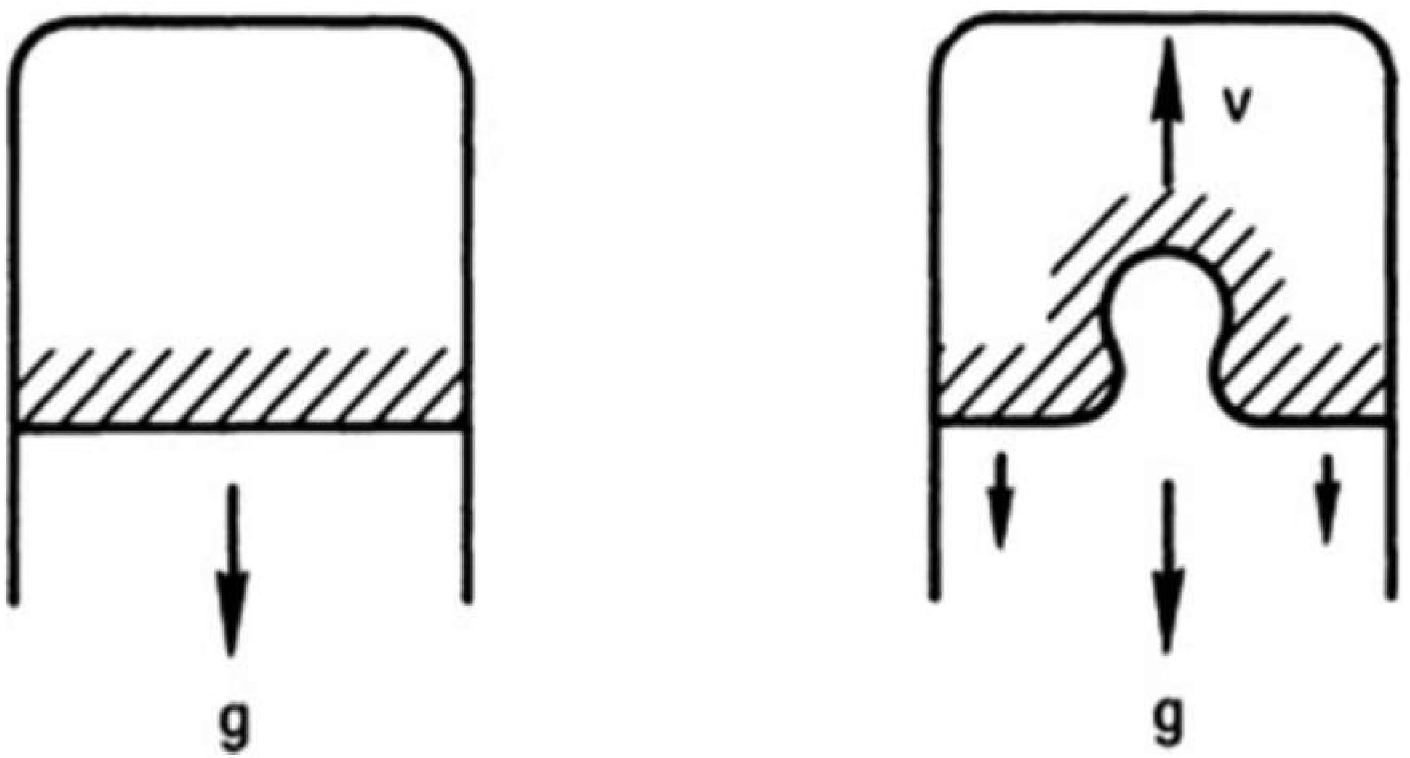
- **Balloning**





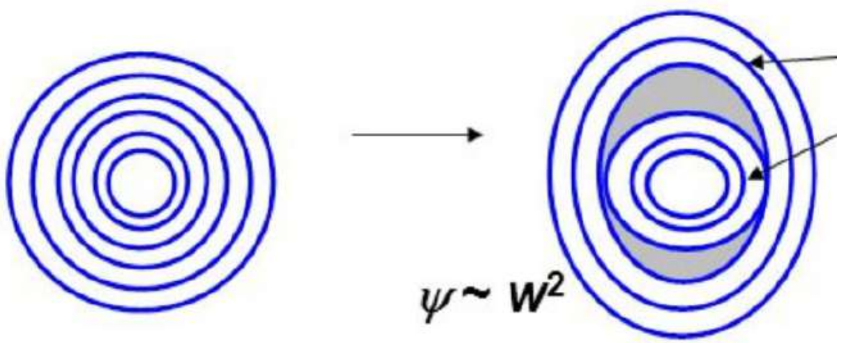
Types of Instabilities, cont.

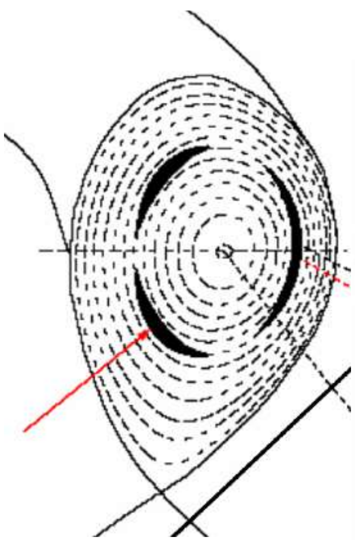
- **Rayleigh-Taylor**

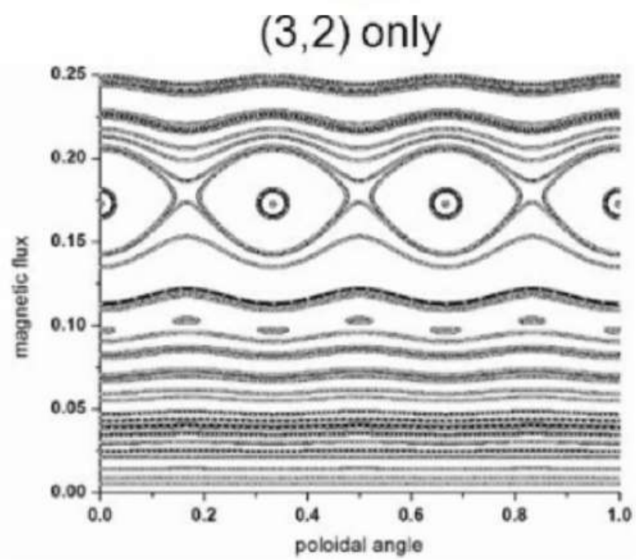


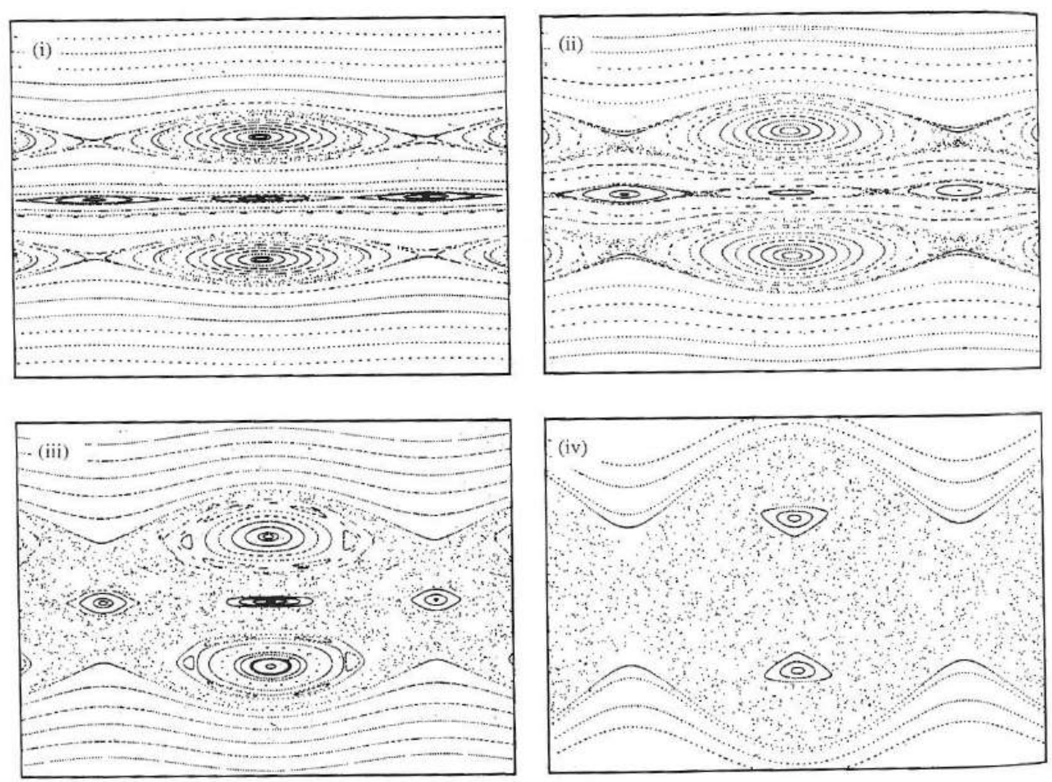
Types of Instabilities, cont.

- **Tearing**









Types of Instabilities, cont.

- **Two Stream Instability**

Two Stream Instability

- **Let us go to the whiteboard**

Two - Stream Instability

Let us we have ions and electrons move with intial velocity V_o, while ions do not have intial velocity

Eq. (1) becomes

$$M n_o \frac{\partial \bar{V}_{i1}}{\partial t} = e n_o \bar{E}_1$$

$n_e = n_o + n_{e1}$

$n_i = n_o + n_{i1}$

$V_e = V_o + V_{e1}$

$V_i = 0 + V_{i1}$

Eq. (2)

$$m n_o \left[\frac{\partial \bar{V}_{e1}}{\partial t} + \left(\bar{V}_o \frac{\partial}{\partial x} \right) \bar{V}_{e1} \right] = -e n_o \bar{E}_1$$

$$\bar{E}_1 = E e^{i(kx - \omega t)} \hat{x}$$

<u>Eq (1)</u>

$$-i\omega M n_o \bar{V}_{i1} = e n_o \bar{E}_1$$

$$\bar{V}_{i1} = \frac{ie}{M\omega} E \hat{x}$$

Eq. (2)

$$mn_0(-i\omega + ikV_0)\bar{V}_{e1} = -en_0\bar{E}_1$$

$$\bar{V}_{e1} = \frac{-ie}{m}\frac{E\hat{x}}{\omega - kV_0}$$

We need another eq. for V_{e1}, V_{i1} and $E\hat{x}$ to use Gaus Law

$$\varepsilon_0 \nabla \cdot E_1 = e(n_{i1} - n_{e1}) \quad (5)$$

So, we use continuity eq. for ions

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x} n_i V_i = 0$$

$$\frac{\partial n_{i1}}{\partial t} + n_0 \frac{\partial}{\partial x} V_{i1} = 0$$

$$n_{i1} = \frac{k}{\omega} n_0 V_{i1} = \frac{ien_0 k}{M\omega^2} E \quad (5)$$

Now, continuity eq. for electrons

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x} n_e V_e = 0$$

$$\frac{\partial n_{e1}}{\partial t} + n_0 \frac{\partial}{\partial x} V_{e1} + V_0 \frac{\partial}{\partial x} n_{e1} = 0$$

$$(-i\omega + ikV_0) n_{e1} + ikn_0 V_{e1} = 0$$

$$n_{e1} = \frac{kn_0}{\omega - kV_0} V_{e1}$$

$$n_{e1} = \frac{iekn_0}{m(\omega - kV_0)} E \quad \text{(7)}$$

Eq. (5)

$$\varepsilon_0 \frac{\partial}{\partial x} E_1 = e(n_{i1} - n_{e1})$$

$$ik\varepsilon_0 E = e(ien_0 kE)\left[\frac{1}{M\omega^2} + \frac{1}{m(\omega - kV_0)^2}\right]$$

$$1 = \frac{e^2 n_0}{\varepsilon_0 m}\left[\frac{m/M}{\omega^2} + \frac{1}{(\omega - kV_0)^2}\right]$$

$$1 = \omega_p^2 \left[\frac{m/M}{\omega^2} + \frac{1}{(\omega - kV_0)^2} \right]$$

$$1 = \frac{m/M}{\left(\frac{\omega}{\omega_p}\right)^2} + \frac{1}{\left(\frac{\omega}{\omega_p} - \frac{kV_0}{\omega_p}\right)^2}$$

$$1 = \frac{m/M}{X^2} + \frac{1}{(X - Y)^2} \equiv F(X, Y) \longrightarrow (8)$$

Solving last eq. we have an overview about the plasma instability. Note that;

① Eq. 8 has four roots

② Finding these roots ⟹ instability

③ These roots can be real or imaginary

④ Why this is important

$$E_1 = E\, e^{i(kx-\omega t)}$$

Solving eq. 8 for ω, we have the following :

$\omega \longrightarrow +$ Real

$\omega \longrightarrow -$ Real

$\omega \longrightarrow +$ Im

$\omega \longrightarrow -$ Im

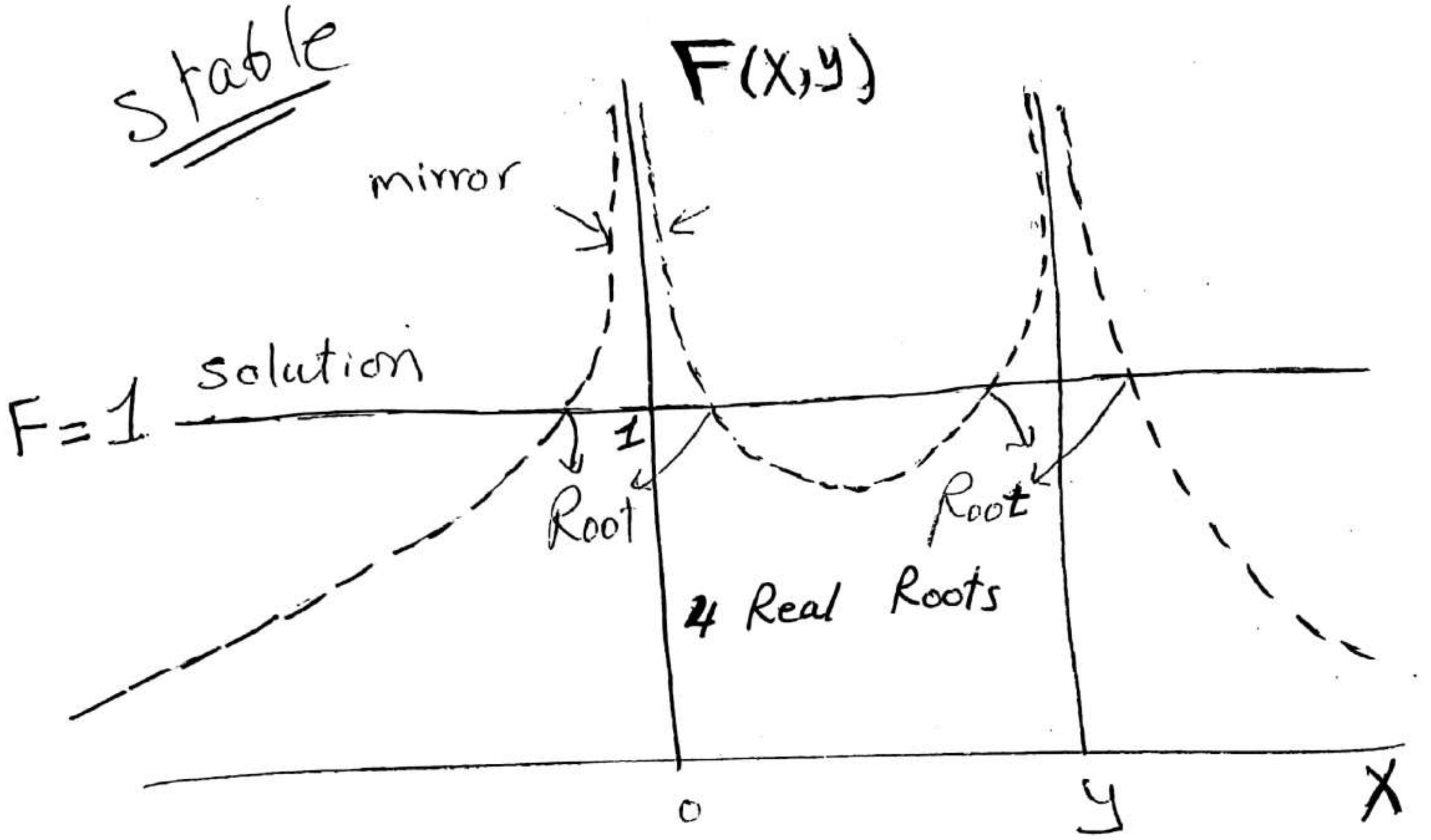
$$E_1 = E\, e^{ikx}\, e^{-i\omega t}$$

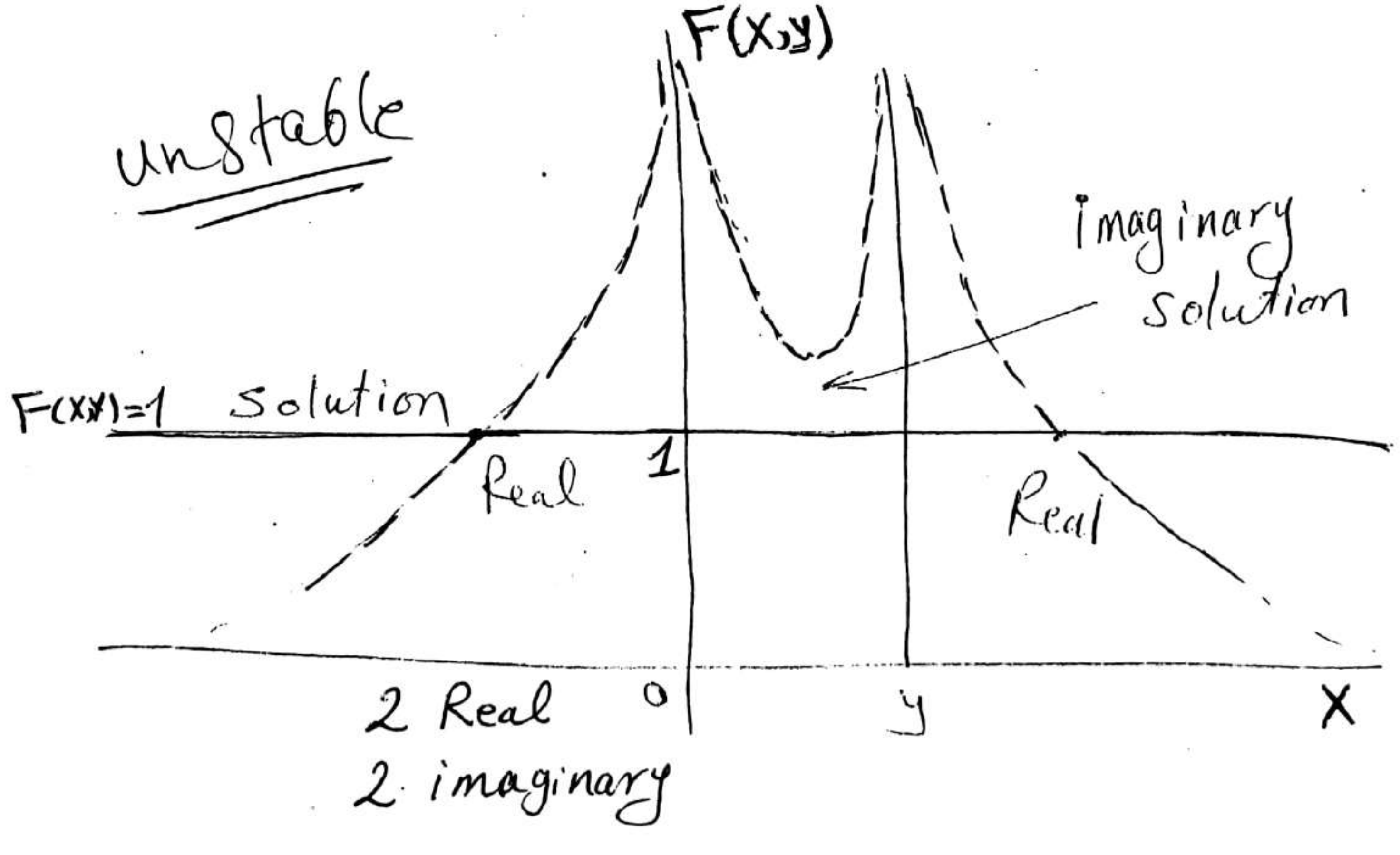
$e^{ikx} \rightarrow$ osci.

$e^{-i\omega t} \rightarrow$

- + Real $\rightarrow$ Osci. $e^{-i\omega_R t}$
- -ve Real $\rightarrow$ Osci $e^{i\omega_R t}$
- + Im $\rightarrow$ growing $e^{\omega_i t}$
- − Im $\rightarrow$ damping $e^{-\omega_i t}$

$$1 = \frac{m/M}{X^2} + \frac{1}{(X-y)^2} \equiv F(X,y)$$





1- Show that the dispersion relation for electrostatic oscillation in a plasma with n beams can be written as

$$1 = 4\pi e^2 \sum_j \frac{N_j Z_j^2}{m_j (\omega - K V_{oj})^2}$$

where N_j, m_j, V_{oj} represent the density mass and Streaming velocity for the jth beam.

2- Find the solution for the dispersion relation given by in problem 1 for the case of two electron beams of equal strength. Show that the maximum growth rate of the instability is found to be $Im(\omega) = \frac{2\omega_t}{\sqrt{2}}$

where

$$\omega_t^2 = 4\pi e^2 \left(\frac{N_e Z_e^2}{m_e}\right)$$

$$= 4\pi e^2 \left[\frac{N_1 Z_1^2}{m_1} \frac{1}{(\omega - kV_{01})^2} + \frac{N_2 Z_2^2}{m_2} \frac{1}{(\omega + kV_{02})^2} \right]$$

for two counterstreaming electron beams

$$N_1 = N_2 = N_e \quad , \quad Z_1 = Z_2 = Z_e$$

$$m_1 = m_2 = m_e \quad , \quad V_{01} = V_{02} = V_0$$

$$1 = \frac{4\pi e^2 N_e Z_e^2}{m_e} \left[\frac{1}{(\omega - kV_0)^2} + \frac{1}{(\omega + kV_0)^2} \right]$$

$$1 = \omega_t^2 \left[\frac{(\omega + kV_0)^2 + (\omega - kV_0)^2}{(\omega - kV_0)^2 (\omega + kV_0)^2} \right]$$

$$(\omega^2 - k^2 V_0^2)^2 = 2\omega_t^2 (\omega^2 + k^2 V_0^2)$$

$$\omega^4 - 2\omega^2 k^2 V_0^2 + k^4 V_0^4 = 2\omega_t^2 \omega^2 + 2\omega_t^2 k^2 V_0^2$$

$$\omega^4 - (2k^2 V_0^2 + 2\omega_t^2)\omega^2 + (k^4 V_0^4 - 2\omega_t^2 k^2 V_0^2)$$

$$\omega^4 - b\omega^2 + c = 0$$

$$\omega^2 = \frac{b \pm \sqrt{b^2 - 4c}}{2}$$

$$\omega = \sqrt{\frac{b \pm \sqrt{b^2 - 4c}}{2}}$$

$$\omega_R + i\omega_i = \sqrt{\frac{b \pm \sqrt{b^2 - 4c}}{2}}$$

To ~~study~~ have ω_i we should have

$$b - \sqrt{b^2 - 4c} < 0$$

$$\sqrt{b^2 - 4c} > b$$

$$b^2 - 4c > b^2$$

$$4c < 0.$$

$$4\left(k^4 V_0^4 - 2\omega_t^2 k^2 V_0^2\right) < 0$$

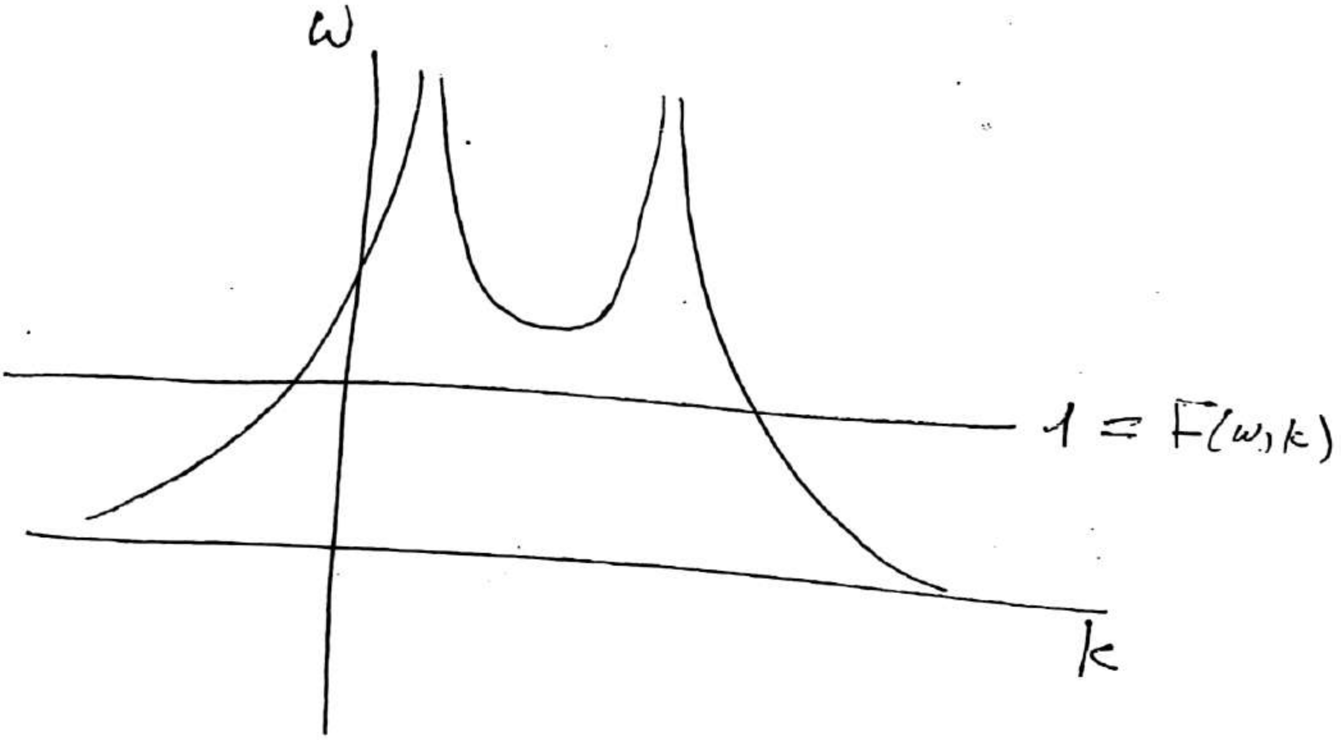
$$4\left(k^2 V_0^2 - 2\omega_t^2\right) < 0$$

for small k (i.e. long wavelength)

$$-8\omega_t^2 < 0 \Rightarrow 8\omega_t^2 > 0$$

$$\omega_i = \sqrt{\frac{8\omega_t^2}{2}} = \sqrt{4\,\omega_t^2}$$

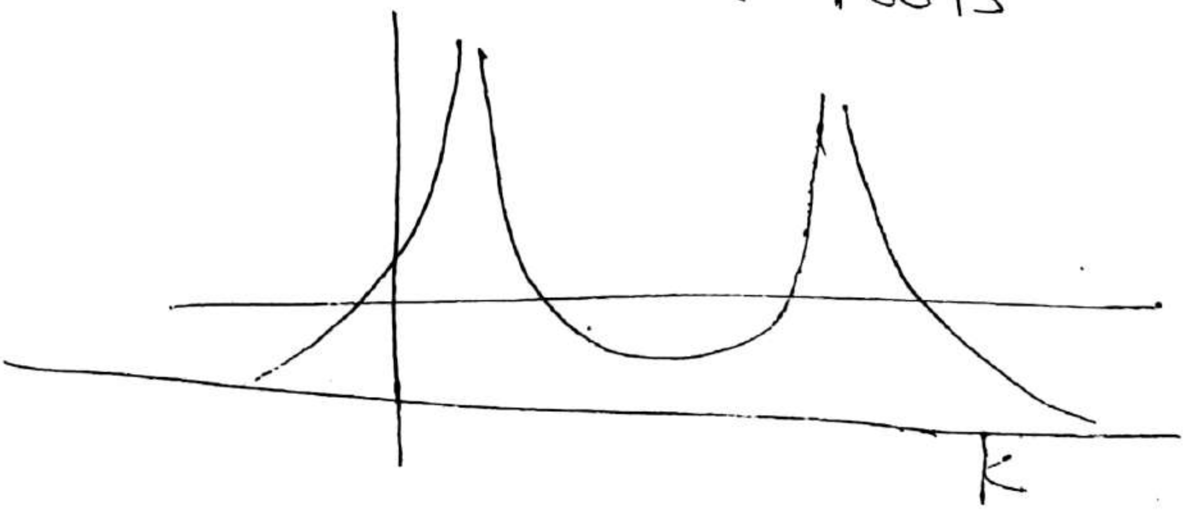
$$\omega_i = 2\,\omega_t$$



For Large K (i.e. small wavelength)

$k^2 V_A^2 - 2\omega_t^2 < 0$ Cannot be satisfied

we have four real roots



Tutorial with Mathematica

- **How to use Mathematica to obtain the following figures...**