Plasma Waves

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Plasma response

When the charge density in the plasma is perturbed by small but finite perturbation where the nonlinear effects are negligible, the plasma responds differently depending on the characteristic frequency of the perturbations. Plasma response to an applied perturbation can be in the form of shielding and oscillations or waves.

- Low frequency (shielding). If the characteristic frequency of the perturbation is low, i.e. $\omega \ll kV_{thi}$, kV_{the}, the plasma electrons, and ions respond isothermally (Maxwellian's distribution function). Then, we obtain the Debye shielding effect as in the preceding section.
- Intermediate frequency (Waves). As the characteristic frequency of the perturbations (ω) increases, the inertia of the charged particles becomes important. If $kV_{thi} \ll \omega \ll kV_{the}$, the electrons and the ions have an inertial adiabatic response .
- High frequency (Oscillations). For high frequency, i.e. $\omega \gg kV_{thi}$, kV_{the}, both electrons, and ions exhibit an inertial or adiabatic response. Then, the plasma responds by oscillating at a collectively determined frequency called the plasma frequency or Langmuir oscillation frequency ω_p.

Wavelike response

- When the perturbation is:
  - Periodic
  - Small amplitude
  - Unbounded
  - Homogenous
  - Time-independent

  the waveform is generally sinusoidal; and there is only one component. This is the situation we shall consider.
- Any sinusoidally oscillating quantity—say, the density n—can be represented as follows:

$$f = f_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{1}$$

  where, in Cartesian coordinates, $\mathbf{k}\cdot\mathbf{r} = k_x\mathbf{x} + k_y\mathbf{y} + k_z\mathbf{z}$. Here f_0 is a constant defining the amplitude of the wave, and $\mathbf{k}$ is called the propagation constant.
- A point of constant phase on the wave moves so that $\frac{d}{dt}(kx - \omega t) = 0$ or

$$\frac{dx}{dt} = \frac{\omega}{k} = v_{ph} \tag{2}$$

  This is called **the phase velocity**. If $\frac{\omega}{k}$ is positive, the wave moves to the right; that is, x increases as t increases, so as to keep $kx - \omega t$ constant. If $\frac{\omega}{k}$ is negative, the wave moves to the left.

Dispersive medium & Dispersion relation

- **The dispersive medium**: The medium in which the velocity depends on frequency and wavenumber.
- A point of constant phase on the wave envelope moves so that $\frac{d}{dt}(\delta kx - \Delta\omega t) = 0$ or

$$\frac{dx}{dt} = \frac{\Delta\omega}{\Delta k} = v_g \tag{3}$$

  This is called **the group velocity**, the velocity with which energy or information can travel.
- **The dispersion relation**: a relationship between the wave number k and the wave frequency ω.

$$v_g = \frac{d}{dk}(kv_{ph}), \tag{4}$$

$$= v_{ph} - \lambda\frac{dv_{ph}}{d\lambda} \tag{5}$$

- **The types of medium**:
  - The normal dispersion: if $\frac{dv_{ph}}{d\lambda} > 0$, $v_g < v_{ph}$.
  - The anomalous dispersion: if $\frac{dv_{ph}}{d\lambda} < 0$, $v_g > v_{ph}$.
  - The nondispersive: if $\frac{dv_{ph}}{d\lambda} = 0$, $v_g = v_{ph}$.

Waves in plasma

- **Electrostatic Waves**:
  - Longitudinal: $\mathbf{E}//\mathbf{k}$.
  - Zero current density: $\mathbf{J} =$.
  - Unperturbed magnetic field: $\mathbf{B} = 0$.
  - Plasma beta: $\beta = \frac{P}{\frac{B^2}{2\mu_0}} \ll 1$.
  - The wave action (i.e., the generation of one quantity by the other and vice versa) in such waves is between the fluid velocity $\mathbf{u}$ and electric field $\mathbf{E}$.

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- **Hydrodynamic Waves**:
  - Longitudinal and Transverse.
  - Ideal MHD: Infinite conductivity $\sigma \rightarrow \infty$.
  - Infinite current density: $\mathbf{J} \rightarrow \infty$.
  - Zero electric field $\mathbf{E} = 0$.
  - magnetized $\mathbf{B}_0 \neq 0$.
  - Low frequency wave: $\omega \ll \omega_{ci}$, Inertialess electrons
  - The wave action: is between the fluid velocity $\mathbf{u}$ and electric field $\mathbf{B}$.

Response model

- **Complete set of Multiple-fluid equations**:

$$\frac{\partial N_s}{\partial t} + \nabla \cdot [N_s \mathbf{u}_s] = 0, \tag{6}$$

$$m_s N_s \left[\frac{\partial}{\partial t} + \mathbf{u}_s \cdot \nabla \right] \mathbf{u}_s = q_s N_s \Big(\mathbf{E} + \mathbf{u}_s \times \mathbf{B} \Big) - \nabla P_s + \nabla \cdot \mathbf{\Pi}_s + \mathbf{R}_{ij} \tag{7}$$

- For self-consistent treatment of a problem, the electric $\mathbf{E}$ and magnetic field $\mathbf{B}$ are determined by using Maxwell's equations:

$$\text{Gauss' Law} \quad \nabla \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0}, \tag{8}$$

$$\text{Gauss' Law} \quad \nabla \cdot \mathbf{B} = 0, \tag{9}$$

$$\text{Faraday's Law} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{10}$$

$$\text{Ampére's Law} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{11}$$

where

$$\text{The charge density} \quad \rho_q = \sum q_s N_s, \tag{12}$$

$$\text{The current density} \quad \mathbf{J} = \sum q_s N_s \mathbf{u}_s \tag{13}$$

Electromagnetic waves

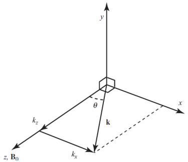
- Transverse perturbation: $\mathbf{E} \perp \mathbf{B} \perp \mathbf{k}$.
- The wave action: is between the fluid velocity **E** and electric field **B**.
- High frequency: Electrons are inertial, Ions are stationary.
- From Maxell's Eqs, we get:

$$\mathbf{k}(\mathbf{k}.\mathbf{E}) - k^2\mathbf{E} + \frac{\omega^2}{c^2}\overleftrightarrow{\epsilon}_p.\mathbf{E} = 0 \qquad (14)$$

where $\overleftrightarrow{\epsilon}_p.\mathbf{E} = \frac{\mathbf{J}}{j\omega\epsilon_0} + \mathbf{E}$.

- The general dispersion relation (Appleton–Hartree equation):

$$\mu^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{(\omega_p^2/\omega_c^2)}{1 - \frac{\omega_c^2 Sin^2\theta}{2(\omega^2-\omega_p^2)} \pm \left[\left(\frac{\omega_c^2 Sin^2\theta}{2(\omega^2-\omega_p^2)}\right)^2 + \frac{\omega_c^2}{\omega^2}Cos^2\theta\right]^{1/2}} \qquad (15)$$



Electromagnetic waves

- Parallel propagation ($\theta = 0$) or along B_0: The disperson relation becomes

$$\mu^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_c)} \tag{16}$$

- This dispersion relation has two branches (+,-).
- The lower (-) branch:

$$\mu^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega(\omega - \omega_c)} \qquad (R\ wave) \tag{17}$$

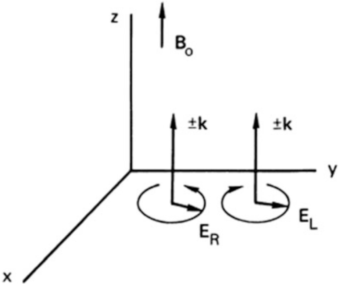
  It is called the whistler mode, the electron-cyclotron mode and the right hand circularly polarized mode.
- The upper (+) branch:

$$\mu^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega(\omega + \omega_c)}, \qquad (L\ wave) \tag{18}$$

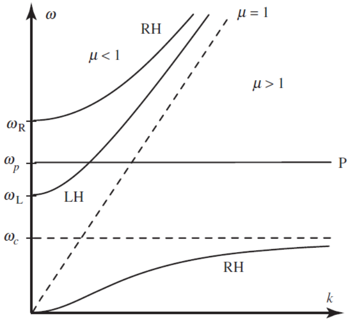
  It is called the left hand circularly polarized mode.
- The electric field vector for the R wave rotates clockwise in time as viewed along the direction of B_0, and vice versa for the L wave.

Electromagnetic waves

- A cutoff occurs in a plasma when the index of refraction goes to zero; that is, when the wavelength becomes infinite, since $\approx \mu = \frac{kc}{\omega}$.
- A resonance occurs when the index of refraction becomes infinite; that is, when the wavelength becomes zero.
- Cuttoffs ($k \to 0$) & Resonances ($k \to \infty$).



(a)



(b)

Electromagnetic waves

- Perpendicular propagation ($\theta = \pi/2$) or across B_0: The disperson relation becomes

$$\mu^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{(\omega_p^2/\omega_c^2)}{1 - \frac{\omega_c^2 Sin^2\theta}{2(\omega^2-\omega_p^2)} \pm \frac{\omega_c^2 Sin^2\theta}{2(\omega^2-\omega_p^2)}} \tag{19}$$

- The upper (+) branch ($\mathbf{E}//\mathbf{B}_0$):

$$\omega^2 = \omega_p^2 + c^2k^2 \qquad (O\ wave) \tag{20}$$

- The mode is called the ordinary mode, since its propagation is not affected by the magnetic field. This means that this wave has the same properties as the transverse electromagnetic wave in a non-magnetized plasma ($\mathbf{B}_0 = 0$) with $\mathbf{u}//\mathbf{E}$.
- The lower (-) branch ($\mathbf{E} \perp \mathbf{B}_0$):

$$\mu^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_H^2} \tag{21}$$

  where $\omega_H^2 = \omega_p^2 + \omega_c^2$ is called is the upper hybrid frequency.
- The mode is called the extraordinary mode. The electric field for this mode is perpendicular to B_0, with E_x and E_y coupled together.

Electromagnetic waves

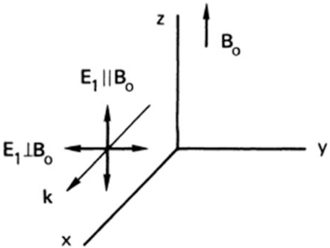
- Cuttoffs ($k \to 0$), we get

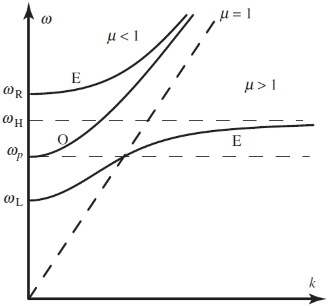
$$\omega_R = \frac{1}{2}\left[(\omega_c^2 + 4\omega_p^2)^{1/2} + \omega_c\right] \tag{22}$$

$$\omega_L = \frac{1}{2}\left[(\omega_c^2 + 4\omega_p^2)^{1/2}\omega_c\right] \tag{23}$$

- & Resonances ($k \to \infty$), we get

$$\omega^2 = \omega_H^2 + \omega_c^2 + \omega_p^2. \tag{24}$$





Hydrodynamic waves

- **Hydrodynamic Waves**:
  - Longitudinal and Transverse.
  - Ideal MHD: Infinite conductivity $\sigma \to \infty$.
  - Infinite current density: $\mathbf{J} \to \infty$.
  - Zero electric field $\mathbf{E}' = \mathbf{E} + \mathbf{u}_m \times \mathbf{B} = 0$.
  - magnetized $\mathbf{B}_0 \neq 0$.
  - Low frequency wave: $\omega \ll \omega_{ci}$, Inertialess electrons
  - The wave action: is between the fluid velocity $\mathbf{u}$ and electric field $\mathbf{B}$.
- our desired dispersion relation (the plasma response to the perturbation) is given by:
- **Linearized ideal MHD equations**:

$$\frac{\partial \rho_m}{\partial t} + \rho_{m0} \nabla \cdot [\mathbf{u}_m] = 0, \tag{25}$$

$$\rho_{m0} \frac{\partial u_m}{\partial t} = -\nabla (P + \frac{\mathbf{B}_0 \cdot \mathbf{B}}{\mu_0}) + \frac{(\mathbf{B}_0 \cdot \nabla)\mathbf{B}}{\mu_0} \tag{26}$$

- For self-consistent, Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \tag{27}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u}_m \times \mathbf{B}_0), \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{28}$$

Hydrodynamic waves

- The perturbed quantities can given by:

$$\rho_m = \rho_0 \frac{\mathbf{k} \cdot \mathbf{u}_m}{\omega} \tag{34}$$

$$P = \Gamma P_0 \frac{\mathbf{k} \cdot \mathbf{u}_m}{\omega} \tag{35}$$

$$B = \frac{(\mathbf{k} \cdot \mathbf{u}_m)\mathbf{B}_0 - (\mathbf{k} \cdot \mathbf{B}_0)_m}{\omega} \tag{36}$$

- our desired dispersion relation (the plasma response to the perturbation) is given by:

$$(\omega^2 - k c_A^2 \cos^2(\theta)) \left[\omega^4 - \omega^2 k^2 (c_A^2 + c_s^2) + k^4 c_A^2 c_s^2 \cos^2(\theta) \right] = 0 \tag{37}$$

which clearly has the roots from highest to lowest frequencies

$$\omega_1^2 = k^2 V_+^2 \approx 10^6/s, \tag{38}$$

$$\omega_2^2 = c_A^2 k^2 \cos^2(\theta) \approx 10^5/s, \tag{39}$$

$$\omega_3^2 = k^2 V_-^2 \approx 10^4/s. \tag{40}$$

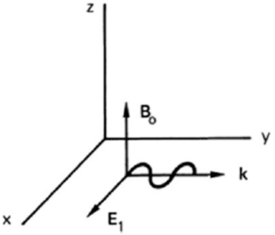
where $V_\pm = \left(\frac{1}{2} \left[c_A^2 + c_s^2 \pm \sqrt{(c_A^2 + c_s^2) - c_A^2 c_s^2 \cos^2(\theta)} \right] \right)^{1/2}$

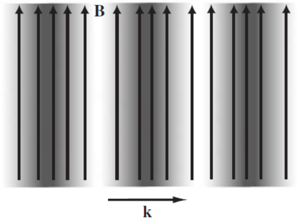
Hydrodynamic waves

- Fast magnetosonic (compressional Alfven) waves ($\mathbf{k} \perp \mathbf{B}_0$): the dispersion relation

$$\omega_1^2 = k^2 V_+^2. \tag{41}$$

- The acoustic speed $c_s = \sqrt{\frac{\Gamma P_0}{\rho_0}} = \sqrt{\frac{\Gamma(T_e+T_i)}{m_i}}$.
- The Alfven speed $c_A = \frac{B_0}{\sqrt{\mu_0\rho_0}}$.
- Compressible $\nabla \cdot \mathbf{u} \neq 0$.
- Perpendicular ($\mathbf{k} \perp \mathbf{B}_0$).
- The perturbation compress or expand the density of the magnetic field lines or the magnetic field strength.
- Simultaneously, they compress or expand the pressure.



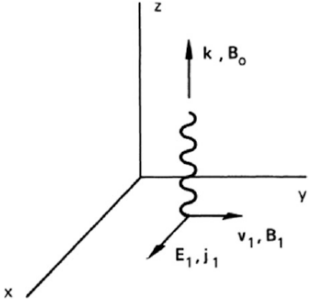


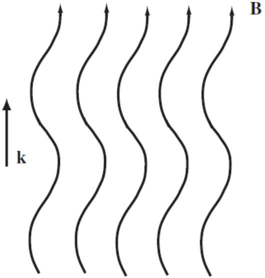
Hydrodynamic waves

- Shear or torsional waves (arbitrary $k_{||}$ and $k_{\perp}$): the dispersion relation

$$\omega_2^2 = c_A^2 k^2 \cos\theta^2. \tag{42}$$

- $\mathbf{k} \cdot \mathbf{u}_m = 0$ and $\mathbf{k} \cdot \mathbf{B}_0 = 0$
- parallel ($\mathbf{k}//\mathbf{B}_0$).
- The perturbation bends, shear or twist the density of the magnetic field lines but doesn't change the magnetic field strength.
- It doesn't produce pressure or mass density perturbation.



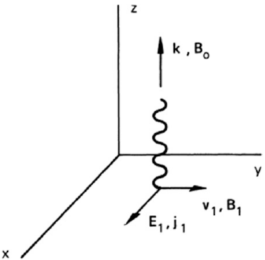


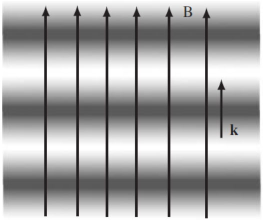
Hydrodynamic waves

- Slow magnetosonic (Acoustic wave) waves ($\mathbf{k} \perp \mathbf{B}_0$): the dispersion relation

$$\omega_3^2 = k^2 V_-^2 \tag{43}$$

- Regular sound waves that propagate "freely" along the magnetic field lines $\mathbf{B}_0$.
- Compressible $\nabla \cdot \mathbf{u} \neq 0$.
- parallel ($\mathbf{k} // \mathbf{B}_0$).
- The perturbation compress $\nabla \cdot \mathbf{u} < 0$ or expand $\nabla \cdot \mathbf{u} > 0$ the pressure or the mass density.
- Doesn't change the magnetic field strength.





Hydrodynamic waves

- The distinction between the fast and slow waves can be further understood by comparing the signs of the wave induced fluctuations in the plasma and magnetic pressures: p and $\mathbf{B}_0 \cdot \mathbf{B}/\mu_0$:

$$\frac{\mathbf{B}_0 \cdot \mathbf{B}}{\mu_0} = \frac{c_A^2}{c_s^2}\left(1 - \frac{c_s^2 \cos^2(\theta)}{V^2}\right) P \tag{44}$$

  where $V = \frac{\omega}{k}$ is the phase velcoity.
- If $V = V_+ > c_S \cos\theta$, we get p and $\mathbf{B}_0 \cdot \mathbf{B}/\mu_0$ have the same sign.
- Thus, in the fast magnetosonic wave the pressure and magnetic energy fluctuations reinforce one another.
- If $V = V_- < c_S \cos\theta$, we get p and $\mathbf{B}_0 \cdot \mathbf{B}/\mu_0$ have the opposite sign.
- Thus, in the slow magnetosonic wave the pressure and magnetic energy fluctuations oppose one another.

Summery

- **Hydrodynamic waves**: Transverse, Low frequency, Ion waves.

  - Along $\mathbf{B}_0$: Shear Alfven and Slow magnetosonic wave.

  - Across $\mathbf{B}_0$: Fast magnetosonic wave.

- **Electromagnetic waves**: Transverse, High frequency, electron waves.

  - Along $\mathbf{B}_0$: right-hand (R) and left-hand (L) circularly polarized wave.

  - Across $\mathbf{B}_0$: plane-polarized wave (O-wave, $\mathbf{E}//\mathbf{B}_0$) and elliptically polarized wave (X-wave, $\mathbf{E} \perp \mathbf{B}_0$).

Thanks for your attention!