Fluctuations and relaxation

in gravitational Systems (and plasmas)

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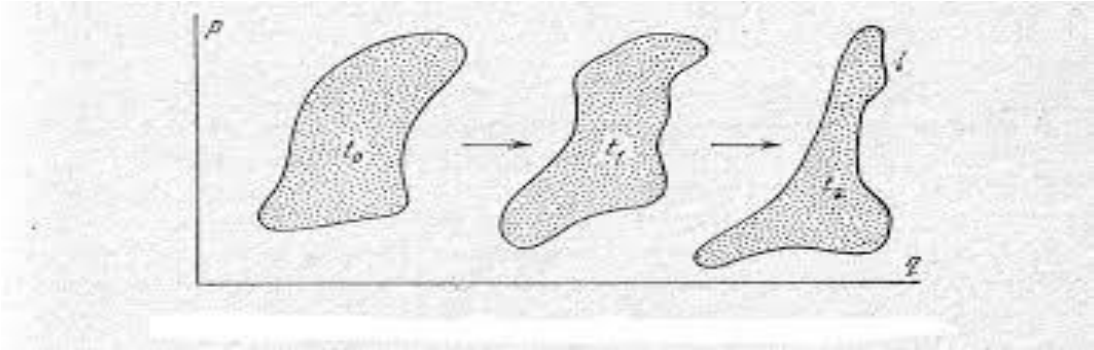
Collisionless System

Conserves 6-d phase space density



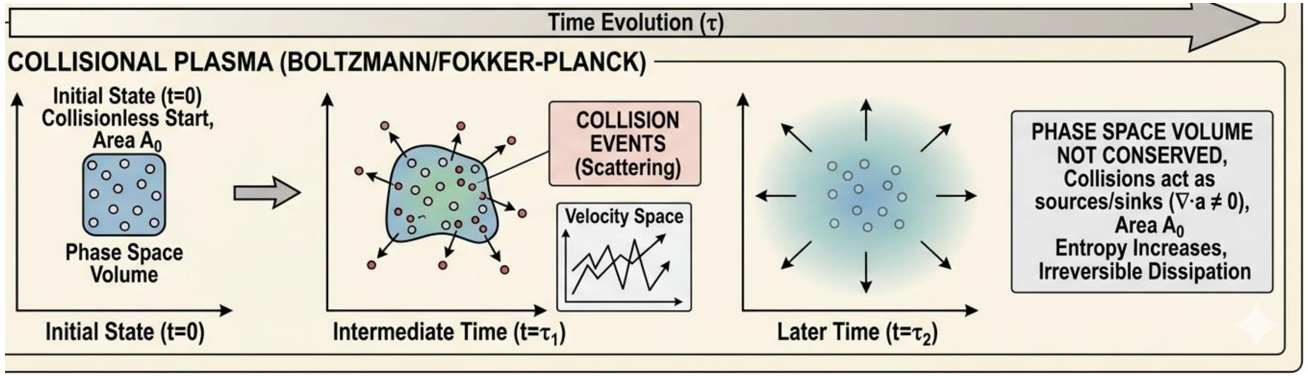
$$\frac{df}{dt} = 0$$

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \dot{\mathbf{v}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

Determined **by mean field**

Collisional

$$\frac{df}{dt} = \left[\frac{\partial f}{\partial t}\right]_{coll}$$



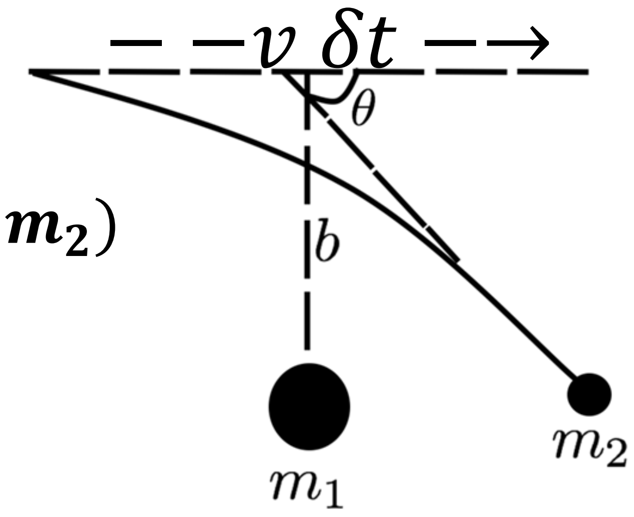
By Google Gemini

Collision can be Coulomb/Grav encounter

- A 'test' star approaches a 'field' star.

- An 'encounter' ($\theta < 90$ deg) with impact parameter b

takes place over **timescale** $\delta t \sim 2\frac{b}{v}$

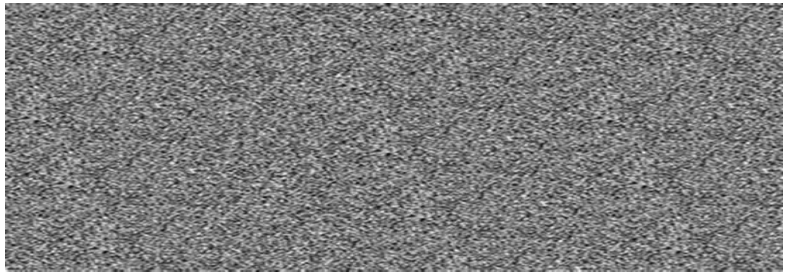


- During which the normal force ~ Gm/b^2 $\quad (m_1 = m_2)$

- So that $\delta v \sim 2\frac{Gm}{b^2}\frac{b}{v} = \frac{2Gm}{bv}$

From Wiki

Many random encounters



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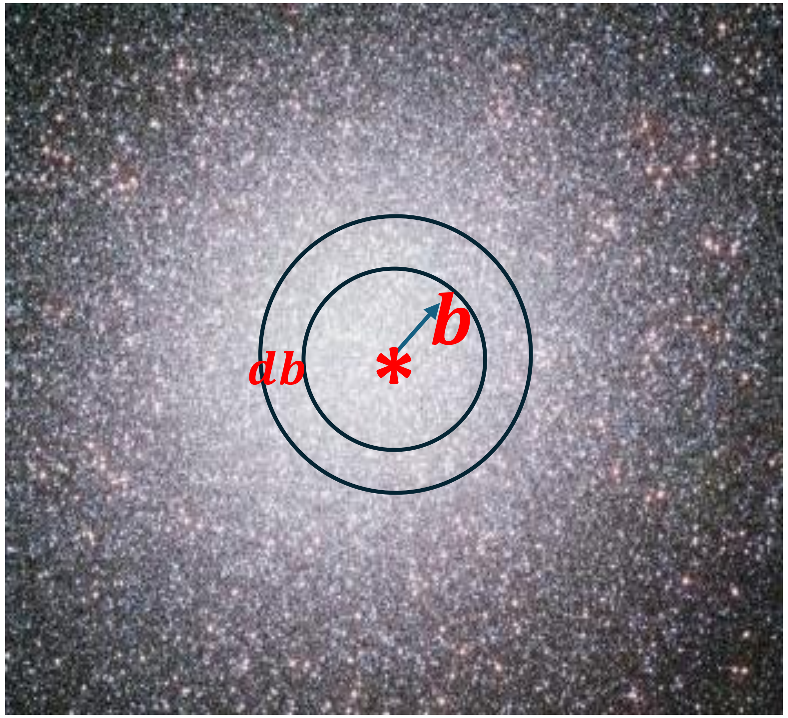


Number of encounters /unit time with

impact parameters $b, \; b + db$

$is \;\; n\, v\, 2\pi b\, db$

The net $\Sigma\, \delta v = 0$

- **Change in squared velocity** after many encounter → **in Diffusion Limit**

$$\sum(\delta v)^2 = \langle(\Delta v)^2\rangle = T\int_{b_{90}}^{b_{\max}} \left(\frac{2Gm}{bv}\right)^2 n\, v \;\; 2\pi b\, db \;=\; \frac{8\pi G^2 n}{v} \;\; \ln(b_{max}/b_{90}) \; * \; T$$

The relaxation time

- The time it takes for $\langle(\Delta v)^2\rangle \sim v^2$

→

$$t_r = \frac{v^3}{8\pi G^2 \rho_0 m \ln \Lambda}$$

Coulomb Logarithm

$$\Lambda = \frac{b_{\max}}{b_{90}}$$

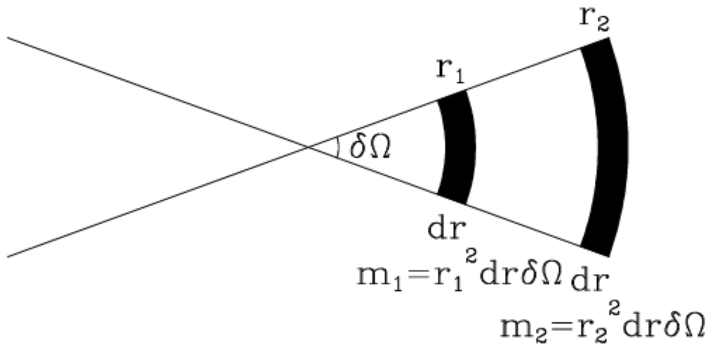
Plasma vs Gravity

$$t_{grav} - \frac{m^2 v^3}{8\,\pi\, G^2 m^4 \;\; n \ln \Lambda}$$

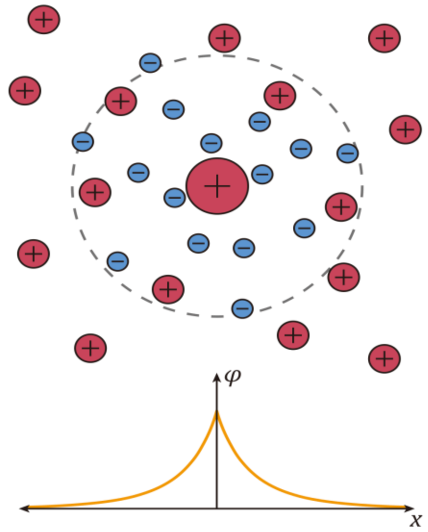
$$Gm^2 \dashrightarrow e^2$$

$$t_{plas} - \frac{m^2 v^3}{8\,\pi\, G^2 e^4 \;\; n \ln \Lambda}$$

Or, in SI units, $Gm^2 \dashrightarrow \dfrac{e^2}{4\pi\epsilon_0}$



b_{max} = system size



b_{max} = Debye length

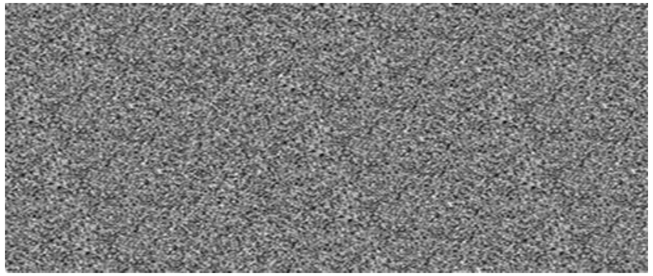
Another Derivation (El-Zant et. al. 2016, 2020)

- **Start with power spectrum of density fluctuations**

$$\delta(\mathbf{r}) = \frac{V}{(2\pi)^3}\int \delta_{\mathbf{k}} e^{-i\mathbf{k}.\mathbf{r}} d\mathbf{k}$$

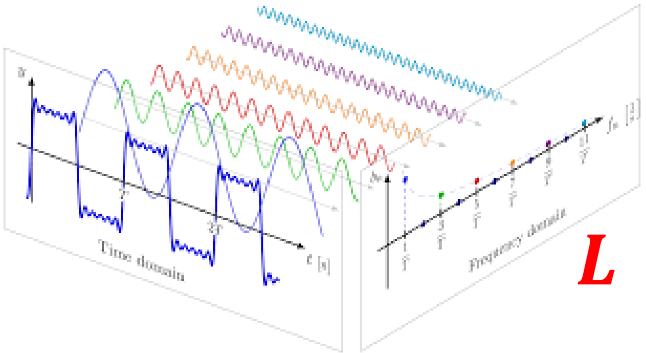
$$\boldsymbol{\delta}\ (\boldsymbol{r}) = \frac{\boldsymbol{\rho}\ (\boldsymbol{r}) - \langle\boldsymbol{\rho}\rangle}{\boldsymbol{\rho}}$$

Density contrast



$$\mathcal{P}(\mathbf{k}) = V\langle|\delta_{\mathbf{k}}|^2\rangle$$

What is a Power Spectrum?



L

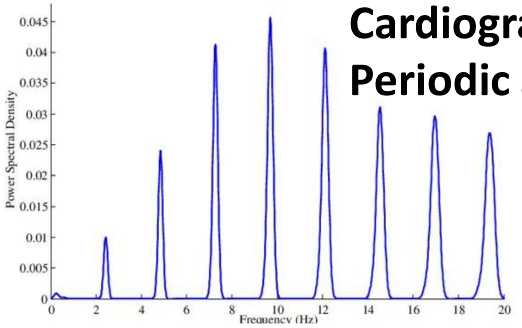
$$k_i = n_i \frac{2\pi}{L}$$

- **Fourier**: **Analyzing** fields on different scales
- ***Function F*** periodic in box L^3 → expand

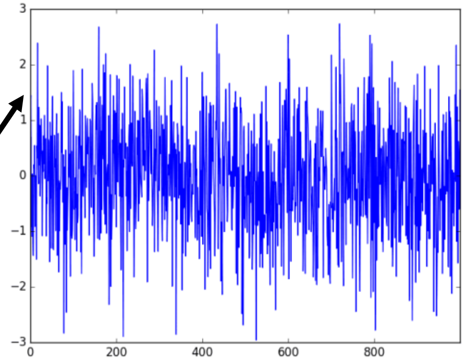
$$F(\mathbf{x}) = \sum F_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}$$

Increase L → number modes increases arbitrarily → continuity

$$F(\boldsymbol{x}) = \left(\frac{L}{2\pi}\right)^3 \int F(\boldsymbol{k}) e^{-i\,\boldsymbol{k}.\boldsymbol{x}}\ d\,\boldsymbol{k}$$



Cardiogram → Periodic Signal



White noise → Flat

Frequency →

Power spectrum ~ F^2

Examples in time-freq domains

From Density to Force fluctuations

- Use Poisson equation $$\nabla^2\Phi = 4\pi G\rho_0\delta$$

- Stationary, homogeneous stochastic process $$\phi_{\mathbf{k}} = -4\pi G\rho_0\delta_{\mathbf{k}}k^{-2}$$

→

- Define force fluctuation power $$\mathcal{P}_F(k) = Vk^2\langle|\phi_k|^2\rangle$$

Fourier Transform → Force Correlation Function

$$\langle \mathbf{F}(0,0).\mathbf{F}(r,t)\rangle = \frac{1}{(2\pi)^3}\int \mathcal{P}_F(k,t)e^{i\mathbf{k}.\mathbf{r}}d\mathbf{k}$$

Look up Wiener Khinchine theorem

$$d\mathbf{v}/dt = \mathbf{F} \Rightarrow \boldsymbol{v} = \int_0^T \boldsymbol{F}\,dt \Rightarrow \langle(\Delta v)^2\rangle = 2\int_0^T (T-t)\langle\mathbf{F}(0).\mathbf{F}(t)\rangle dt$$

Force **random and 'stationary'**; statistics do not depend on time (CF. extra materiel)

$$\langle F(0)F(t)\rangle = \langle F(0)F(r = v_r t)\rangle$$

White Noise and Two Body Relaxation

Delta correlation

→

In diffusion limit

$vt \gg \lambda_{max} \sim b_{max}$

$$\langle \rho(0)\ \rho(\mathbf{r}) \rangle = m\, \rho_0 \delta_D(\mathbf{r}) \longrightarrow \mathcal{P} = \frac{m}{\varrho_0} = \frac{1}{n}$$

$$\langle (\Delta v_p)^2 \rangle = \frac{8\pi G^2 \rho_0 m}{v_r} T \ln \Lambda \longrightarrow t_r = \frac{v^3}{8\pi G^2 \rho_0 m \ln \Lambda}$$

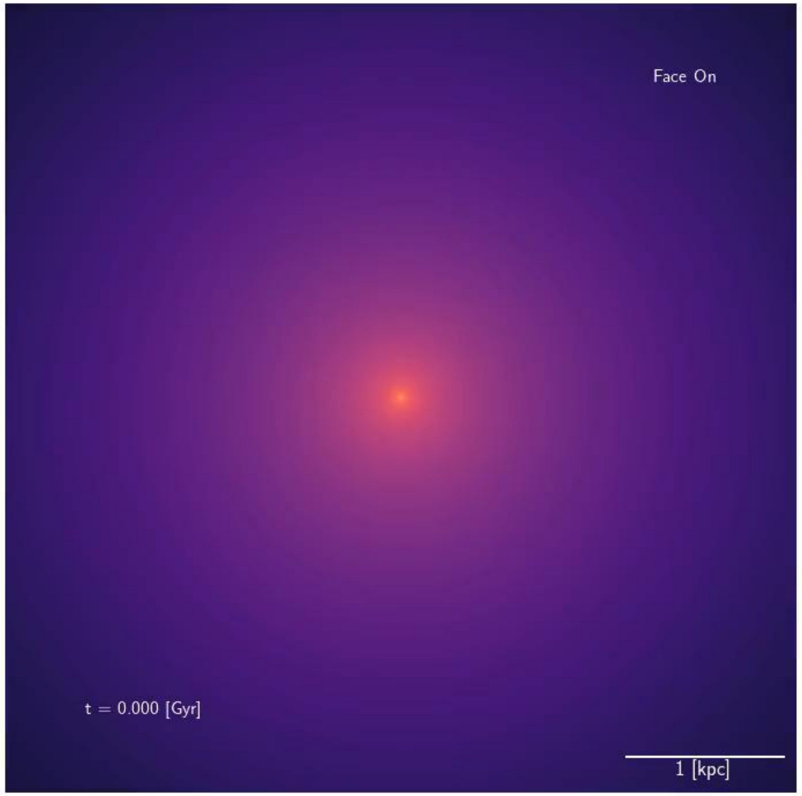
[1908.09061] The effect of fluctuating fuzzy axion haloes on stellar dynamics: a stochastic model Section 2,1

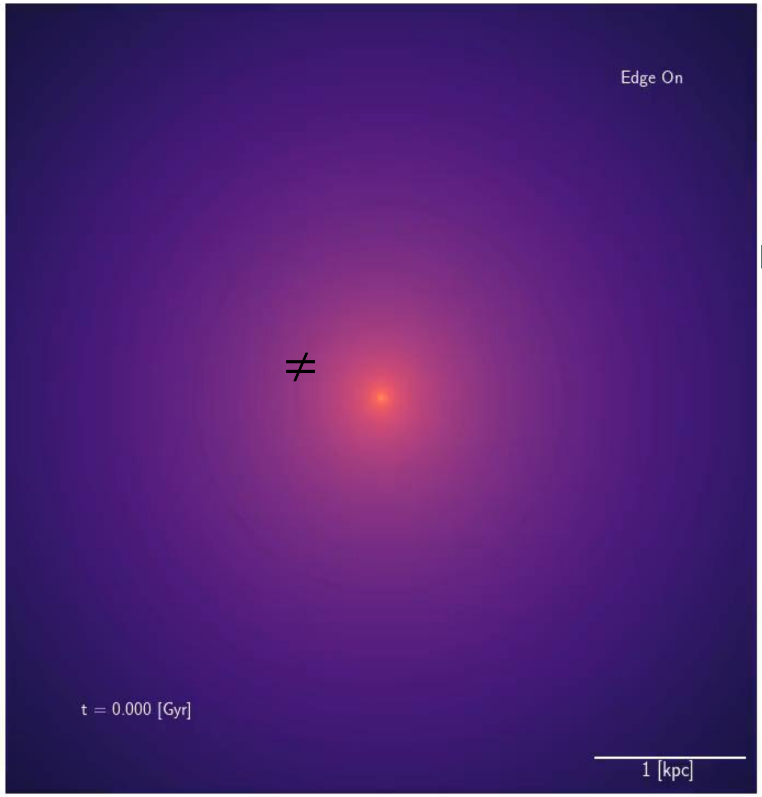
The two relaxation times are equivalent

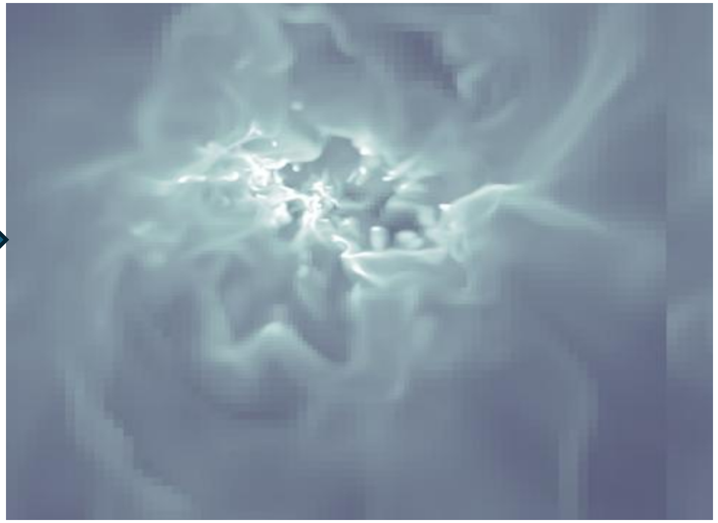
- Latter method generalizable to cases where the ‘noise’ is more complicated than simple white noise.
- Examples where fluctuations cannot be described by white noise, but are nearly power laws, include turbulence --- both compressible and incompressible.
- Turns out important for effect of gas fluctuations in galaxies

Density Fluctuations in Hydro of Galaxy Gas

(Hydrodynamic simulation of gas driven by exploding stras)







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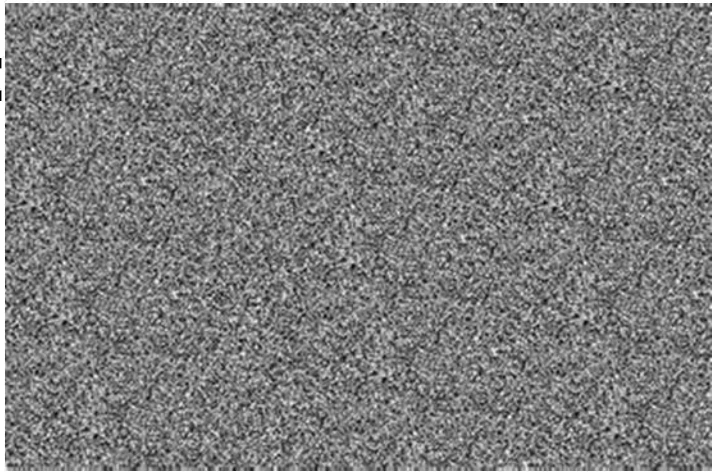
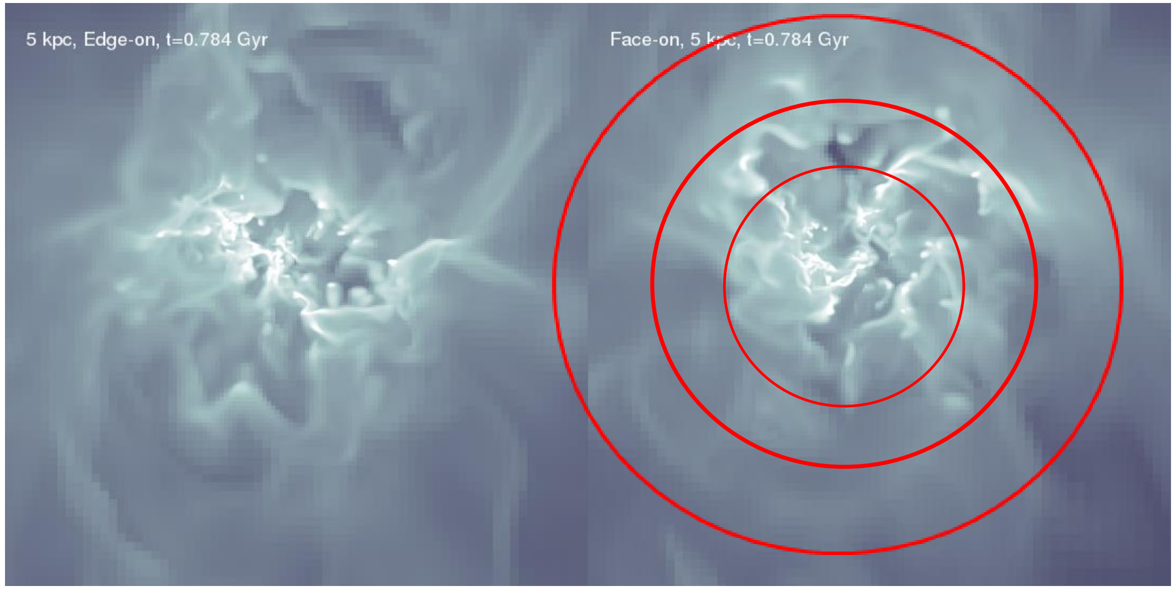


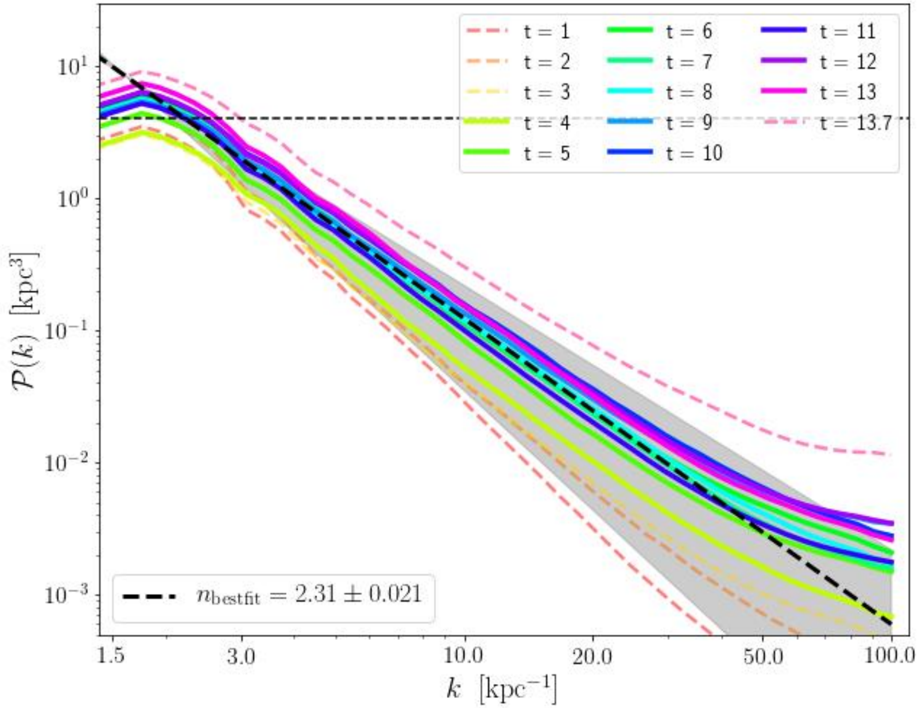
~Stars and dark matter

≈

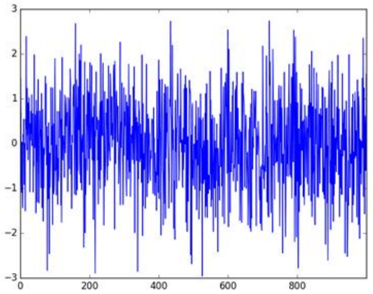
White noise

Mass Fluctuations and Power Spectra (Hashim et. al. 2023):

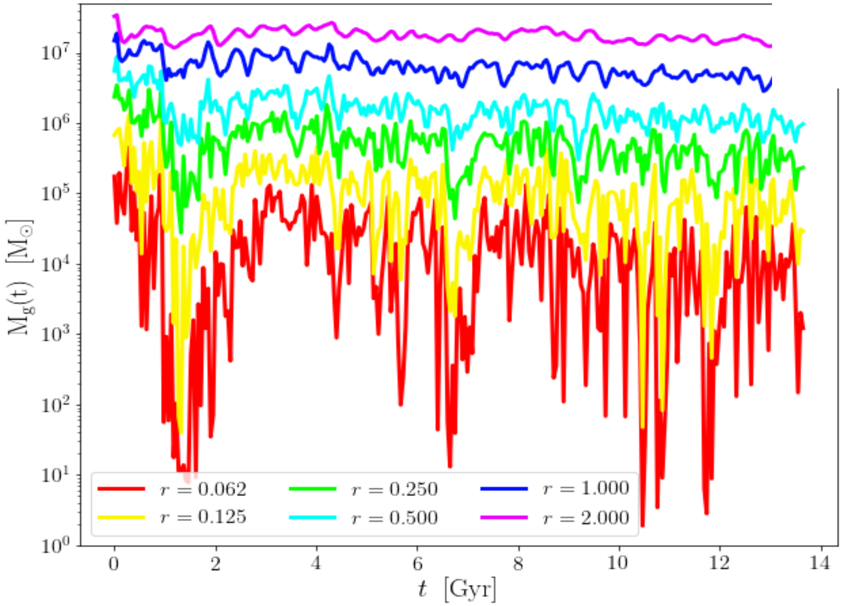




Mass fluctuations



White noise



Power Law Spectrum

$$\sim k^{-n_p} \quad with\, n \sim 2-3$$

Repeat: From Density to Force fluctuations

- Use Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho_0 \delta$$

- Stationary, homogeneous stochastic process

$$\phi_{\mathbf{k}} = -4\pi G \rho_0 \delta_{\mathbf{k}} k^{-2}$$

→

- Define force fluctuation power

$$\mathcal{P}_F(k) = V k^2 \langle |\phi_k|^2 \rangle$$

Relaxation time for power law fluctuations

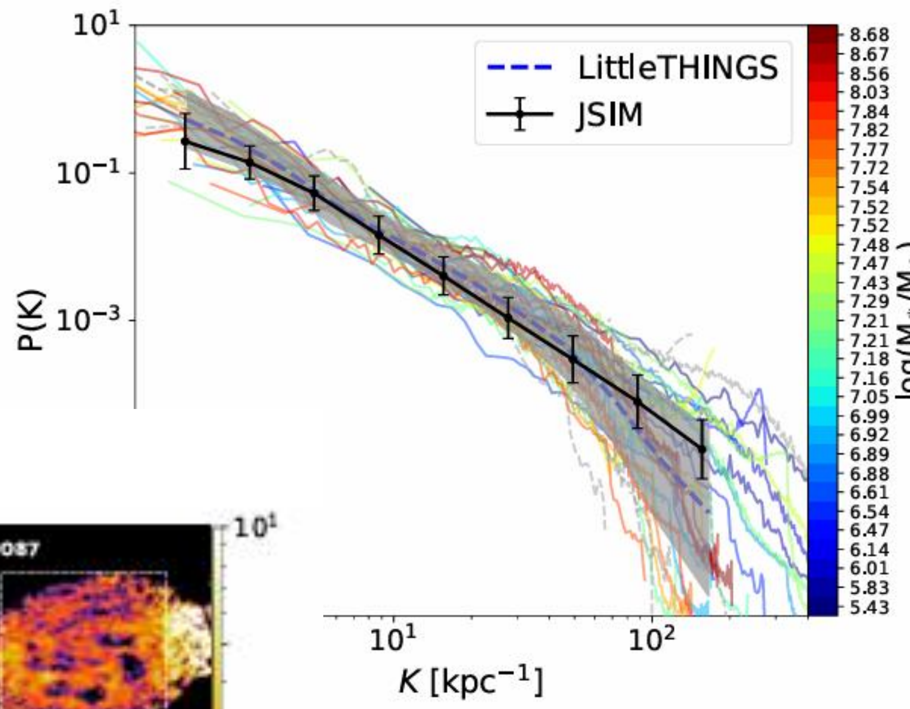
$$\langle(\Delta v)^2\rangle = 2\int_0^T (T-t)\langle \mathbf{F}(0).\mathbf{F}(t)\rangle dt$$

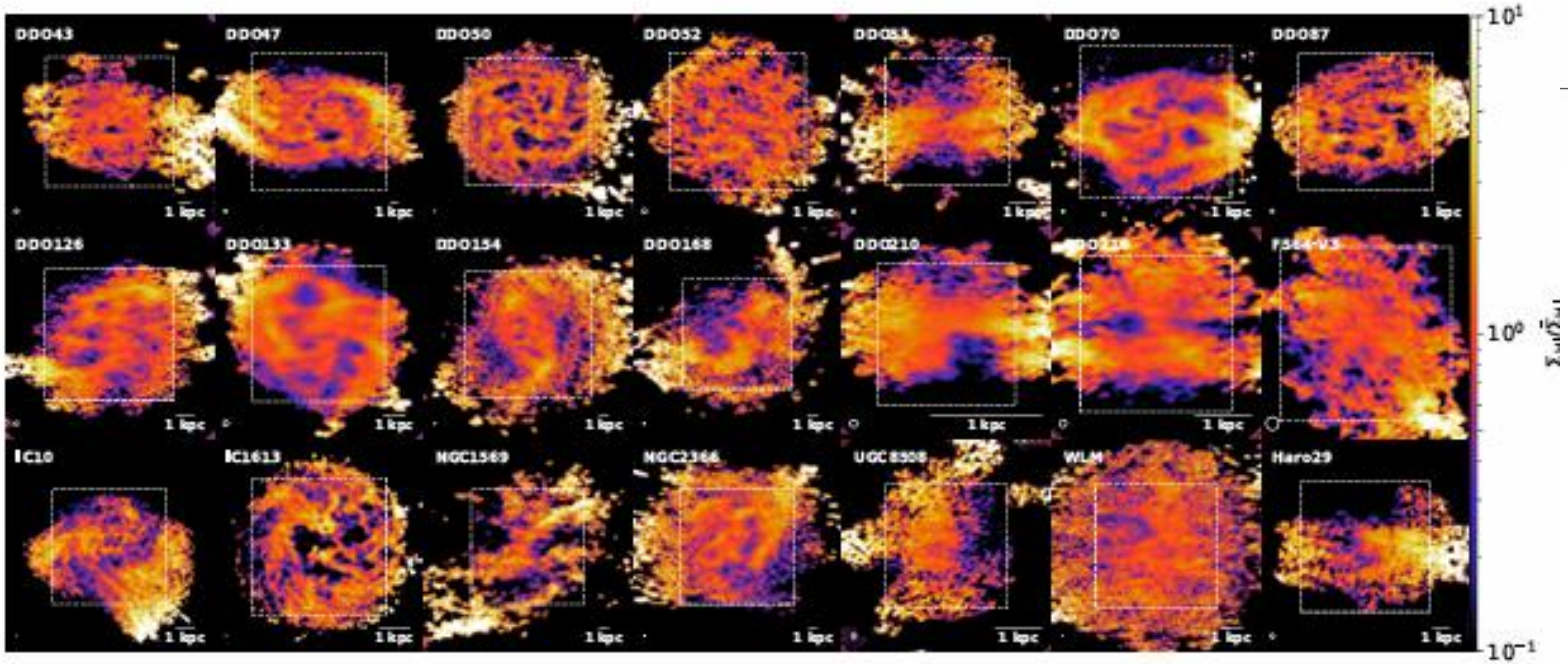
$$\langle(\Delta v)^2\rangle = \frac{8\pi(G\rho_0)^2\mathcal{P}(k_m)}{n v_r}T.$$

El-Zant et. al. (2016)
1908.09061

$$t_r \simeq \frac{n_p v^3}{8\pi G^2 \rho_0^2 \mathcal{P}(k_m)},$$ compare with $$t_r \simeq \frac{n\ \ v^3}{8\pi G^2 \rho_0^2 \ln \Lambda}$$

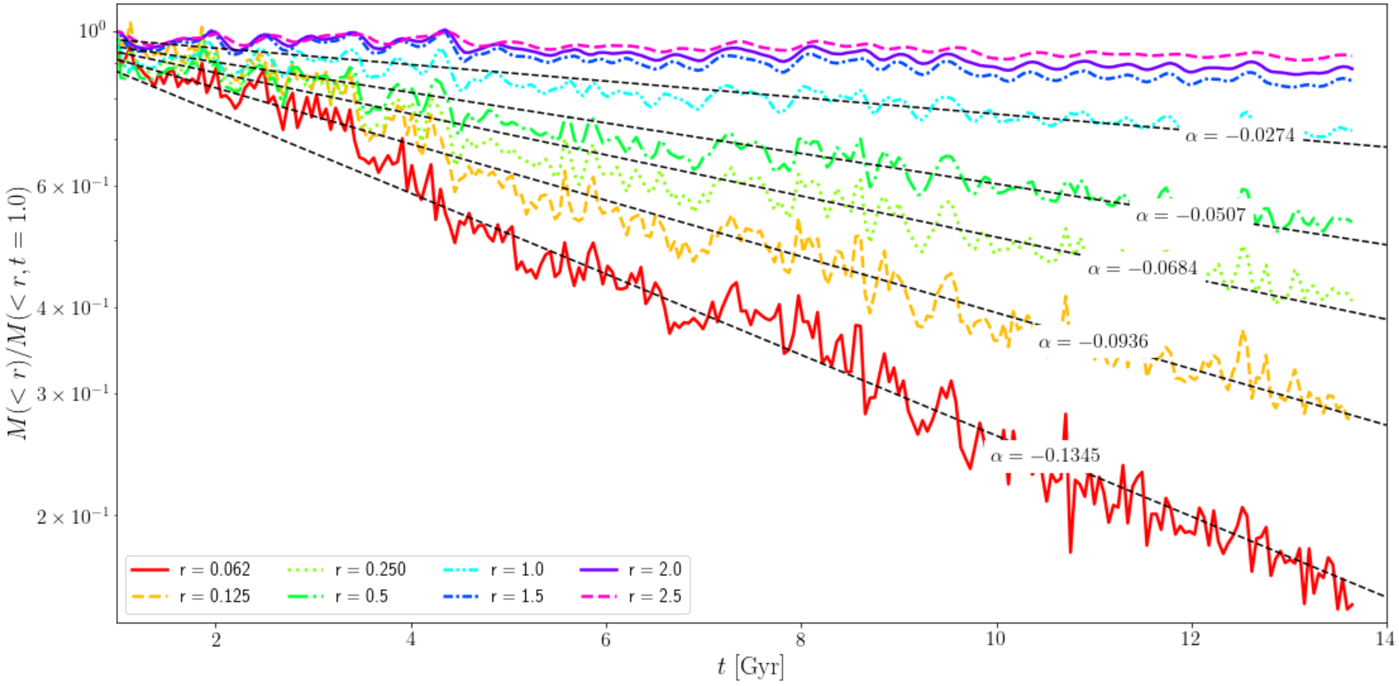
Compare with Observations



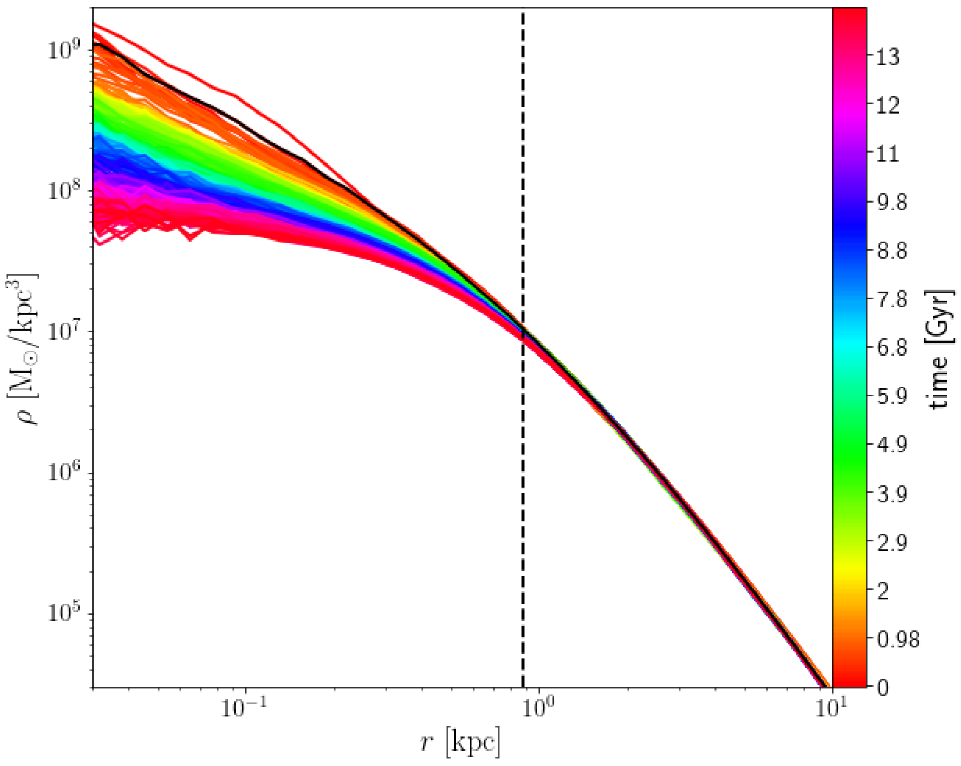


Effect: Mass Migration and Density Reduction (Needs a whole lecture or more…)

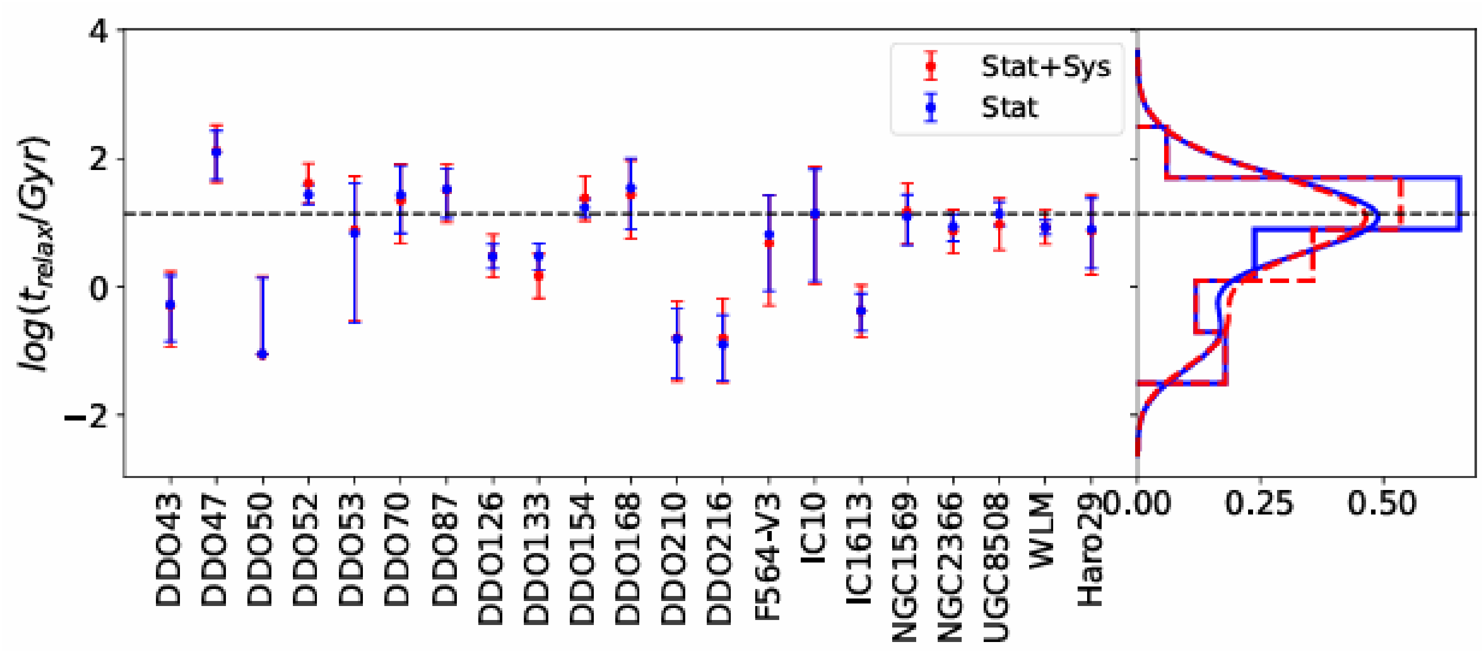
Mass of galaxy within given rad



Density



The Relaxation time for density reduction in our sample of galaxies



Final remarks

- The standard, white noise, two body relaxation is ~ same for plasma and gravity
- In gravity the idea can be generalized to power law spectra appropriate for fully turbulent gas affecting other galactic components
- Can explain some properties of galaxies
- Any interesting analogy in plasma?

Extra Material

Derivation of velocity variance from force correlation

1603.00526 appendix B

Random (Gaussian) force: ensemble average can be taken inside intergrals

Considering the effect of the random perturbation force in direction i during a time T, the equation of motion leads to

$$\frac{dx_i}{dt} = v_{0i} + \int_0^T F_i(\tau) d\tau, \qquad \text{(B1)}$$

where v_{0i} is the initial velocity in direction i. The velocity variance is obtained by averaging this equation:

$$\langle (\Delta v_i)^2 \rangle = \left\langle \left(\frac{dx_i}{dt} - v_{0i} \right)^2 \right\rangle = \int_0^T \int_0^T \langle F_i(\tau) F_i(\tau') \rangle \, d\tau d\tau'. \qquad \text{(B2)}$$

The integrand is symmetrical in τ, τ' and the integration domain correspond to a square of length T in the corresponding plane. We can thus replace the integral over the square by twice the integral over the triangle defined by $0 < \tau < T$ and $\tau < \tau' < T$ so that

$$\langle (\Delta v_i)^2 \rangle = 2 \int_0^T d\tau \int_\tau^T d\tau' \, \langle F_i(\tau) F_i(\tau') \rangle, \qquad \text{(B3)}$$

which can be rewritten as

$$\langle (\Delta v_i)^2 \rangle = 2 \int_0^T dt \int_0^{T-t} d\tau \, \langle F_i(\tau) F_i(\tau + t) \rangle. \qquad \text{(B4)}$$

The perturbations being stationary, $\langle F_i(\tau) F_i(\tau + t) \rangle = \langle F_i(0) F_i(t) \rangle$, the expression simplifies to

$$\langle (\Delta v_i)^2 \rangle = 2 \int_0^T (T - t) \langle F_i(0) F_i(t) \rangle \, dt. \qquad \text{(B5)}$$

and the total velocity variance is given by

$$\langle (\Delta v)^2 \rangle = 2 \int_0^T (T - t) \langle \mathbf{F}(0) . \mathbf{F}(t) \rangle \, dt. \qquad \text{(B6)}$$